

# CONSISTENCY BANDS FOR THE MEAN EXCESS FUNCTION AND APPLICATION TO GRAPHICAL GOODNESS OF FIT TEST FOR FINANCIAL DATA

Gane Samb LO<sup>(1,2)</sup>, Diadie BA<sup>(2)</sup>, Elhadji DEME<sup>(2)</sup> and Cheikh T. SECK<sup>(2,3)</sup>.

<sup>(1)</sup>LSA, Université Pierre et Marie Curie, France.

<sup>(2)</sup>LERSTAD, Université Gaston Berger, Sénégal.

<sup>(3)</sup>Université de Bambey, Sénégal.

**ABSTRACT.** In this paper, we use the modern setting of functional empirical processes and recent techniques on uniform estimation for non parametric objects to derive consistency bands for the mean excess function in the i.i.d. case. We apply our results for modelling financial data, in particular Dow Jones data basis to see how good the Generalized hyperbolic distribution models fit monthly data.

**Keywords:** Mean excess function, Vapnik-Chervonenkis classes, Entropy numbers, Bracketing numbers, Glivenko-Cantelli and Donsker classes, Functional empirical processes, Stochastic processes, Talagrand bounds, Generalized hyperbolic distributions.

## 1. INTRODUCTION

Let  $X$  be a random variable defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , and let  $F$  be its distribution function with endpoint

$$x_F = \sup\{x \in \mathbb{R}, F(x) < 1\},$$

and let  $\bar{F} = 1 - F$  its survival function.

Throughout the paper we suppose that  $\mathbb{E}|X| < \infty$ . The mean excess function  $e(u)$  of  $X$  is defined by (see, e.g., Kotz and Shanbhag [6], Hall and Wellner [5], Guess and Proschan [4])

$$(1.1) \quad e(u) = \mathbb{E}(X - u | X > u) = \begin{cases} \frac{1}{\bar{F}(u)} \int_u^\infty \bar{F}(t) dt & \text{if } \bar{F}(u) > 0 \\ 0 & \text{whenever } \bar{F}(u) = 0. \end{cases}$$

A natural way to estimate the mean excess function  $e(u)$  is achieved by using the plug-in method, that is replacing the survival function in (1.1) by its empirical counterpart, as did Yang [3].

Now consider a sequence  $X_1, X_2, \dots$  of independent copies of  $X$ . The plug-in estimator of  $e(u)$ , for  $n \geq 1$ , is

$$(1.2) \quad e_n(u) = \frac{\sum_{i=1}^n (X_i - u) \mathbb{I}_{[X_i > u]}}{\sum_{i=1}^n \mathbb{I}_{[X_i > u]}} = \frac{\sum_{i=1}^n X_i \mathbb{I}_{[X_i > u]}}{\sum_{i=1}^n \mathbb{I}_{[X_i > u]}} - u,$$

where  $\mathbb{I}_{[X > u]} = 1$  if  $X > u$  and 0 otherwise.

For notation convenience, we denote

$$\mathbb{P}_X(f_u) = \int f_u(x) dF(x) = \int_u x dF(x)$$

and

$$\mathbb{P}_X(g_u) = \int g_u(x) dF(x) = \int_u dF(x) = \bar{F}(u)$$

where  $f_u(x) = x\mathbb{I}_{[x>u]}$ ,  $g_u(x) = \mathbb{I}_{[x>u]}$ , and  $\mathbb{P}_X$  is the probability law of  $X$ .

We also denote by  $\mathbb{P}_n$  the empirical measure associated with the sample  $X_1, \dots, X_n$ . We have

$$\mathbb{P}_n(f_u) = \frac{1}{n} \sum_{i=1}^n X_i \mathbb{I}_{[X_i>u]} \quad \text{and} \quad \mathbb{P}_n(g_u) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{[X_i>u]}.$$

Formulae (1.1) and (1.2) lead to

$$e(u) = \begin{cases} \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} - u & \text{if } u \leq x_F \\ 0 & \text{if } u > x_F. \end{cases}$$

and

$$e_n(u) = \begin{cases} \frac{\mathbb{P}_n(f_u)}{\mathbb{P}_n(g_u)} - u & \text{if } u \leq X_{n,n} \\ 0 & \text{if } u > X_{n,n}, \end{cases}$$

where  $X_{n,n} = \max_{1 \leq i \leq n} X_i$ .

One of the most important motivation of the study of the mean excess function comes from extreme value theory (EVT). Indeed this function  $e(u)$  is linear in the threshold  $u$  when  $F$  is a Generalized Pareto distribution ( $\mathcal{GPD}$ ) and this is quite a powerful graphical test for such distributions.

By using the Vapnik-Chervonenkis (VC) classes and the entropy numbers technics, we have been able to establish that the empirical mean excess function  $e_n(u)$  converges almost surely and uniformly. We showed that for any  $u_1$  less than the upper endpoint of the distribution  $F$ ,

$$\sup_{u \leq u_1} |e_n(u) - e(u)| \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.$$

Next, by using the modern theory of functional empirical process mainly exposed in [10], we proved that the empirical mean excess function  $e_n(u)$  also weakly converges, that is

$$\{\sqrt{n}(e_n(u) - e(u)), u \in I\} \rightarrow^w \{\mathbb{G}(h_u), u \in I\}.$$

where  $\mathbb{G}$  is a Gaussian process and  $\{h_u, u \in I\}$  is a function family to be both precised later.

Furthermore, using Talagrand's inequality (see [8]), and Mason and al. technics (see [9]), we arrived at finding our best achievement: that is finding consistency bands for the mean excess function  $e(u)$ . Precisely we establish that for any interval  $I = [u_0, u_1]$ ,  $u_1$  being less than the upper endpoint of  $F$  and for any  $\varepsilon > 0$ , we have for  $n$  large

$$\mathbb{P}\left(e_n(u) - \frac{E_n}{\sqrt{n}} < e(u) < e_n(u) + \frac{E_n}{\sqrt{n}}, u \in I\right) > 1 - \varepsilon,$$

where  $(E_n)_{(n \geq 1)}$  is a non-random sequence of real numbers precised in **Theorem 3** and where  $F$  satisfies a very slight condition.

These results allowed us to set graphical goodness of fitting test based on the empirical mean excess function and to apply this test to Dow Jones data. We found that the Generalized hyperbolic family distribution reveals, itself, to generally fit financial data.

In this remainder of the text, we are going to detail this outlined results, to demonstrate them, to make simulations studies about them, and finally to apply them to financial data.

The paper is organized as follows. We state uniform almost sure (*a.s*) convergence results in SECTION 2 and finite-distribution and functional normality theorems in SECTION 3.

SECTION 4 is devoted to setting *a.s* consistency bands for the mean excess function. In SECTION 5, simulation studies and data driven applications using Dow Jones data are provided. We finish the paper by a concluding section.

Before we go any further, it is worth mentioning that, in the sequel, all the suprema, taken over  $u < u_1$ , are measurable since the functions of  $u$  that we consider below, are left or right continuous. This means that we are in the pointwise-measurability scheme. Thus, even when we use the results and concepts in [10], we do not need exterior either interior integrals or convergence in outer probability.

## 2. ALMOST SURE CONVERGENCE

In this section we are going to prove the uniform almost sure convergence of the empirical mean excess function by using Vapnik-Chervonenkis (VC) classes and bracketing numbers.

**Theorem 1.** *Suppose that  $\mathbb{E}|g_u(X_1)| < \infty$  and  $\mathbb{E}|f_u(X_1)| < \infty$ , then*

$$\sup_{u < x_F} |\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)| \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty$$

and

$$\sup_{u < x_F} |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.$$

For any fixed  $u_1 < x_F$ , for  $\mathbb{E}X_1^2 < +\infty$

$$\sup_{u \leq u_1} |e_n(u) - e(u)| \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.$$

**Proof.** We observe that  $\mathcal{F}_1 = \{g_u, u < x_F\}$  is a class of monotone real functions with values in  $[0, 1]$ . By Theorem 2.7.5 in [10], the bracketing number  $N_{[]}(\varepsilon, \mathcal{F}_1, L_r(Q))$  is finite (bounded by  $\exp(K/\varepsilon)$ , for every probability measure  $Q$ , any real  $r \geq 1$ , and a constant  $K$  that only depends on  $r$ ). Since  $\mathbb{E}|g_u(X_1)| < \infty$  for  $u < x_F$ ,  $\mathcal{F}_1$  is functional Glivenko-Cantelli class in the sense of Theorem 2.4.1 in [10], meaning that

$$(2.1) \quad \sup_{u < x_F} |\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)| \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.$$

The class  $\mathcal{F}_2 = \{f_u, u \in [u_0, u_1]\}$ , with  $u_1 < x_F$ , is a Vapnik-Chervonenkis class with index  $V(\mathcal{F}_2) = 3$  and its envelop is  $G = \max(|f_{u_0}(x)|, |f_{u_1}(x)|)$ . Then it satisfies the uniform entropy condition 2.4.1 in [10]. Then  $\mathcal{F}_2$  is a Donsker class and hence it is a Glivenko Cantelli class, that is

$$(2.2) \quad \sup_{u < x_F} |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.$$

To finish, fix  $u_1 < x_F$ . Then for  $u \leq u_1$  and  $n$  large enough, we have

$$\begin{aligned} e_n(u) - e(u) &= \frac{\mathbb{P}_n(f_u)}{\mathbb{P}_n(g_u)} - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} \\ &= \frac{\mathbb{P}_n(f_u)}{\mathbb{P}_n(g_u)} - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_n(g_u)} + \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_n(g_u)} - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} \\ &= (\mathbb{P}_n(g_u))^{-1}(\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)) - \mathbb{P}_X(f_u) \times \frac{\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)}{\mathbb{P}_n(g_u)\mathbb{P}_X(g_u)}. \end{aligned}$$

Then

$$(2.3) \quad |e_n(u) - e(u)| \leq |\mathbb{P}_n(g_u)|^{-1} \times |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| + |\mathbb{P}_X(f_u)| \times \frac{|\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)|}{|\mathbb{P}_n(g_u)\mathbb{P}_X(g_u)|}.$$

Let

$$(2.4) \quad \epsilon_n = \sup_{u < x_F} |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| \quad \text{and} \quad \delta_n = \sup_{u < x_F} |\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)|.$$

From (2.1) and (2.2) above, we have

$$\epsilon_n \rightarrow 0 \quad \text{a.s.} \quad \text{and} \quad \delta_n \rightarrow 0 \quad \text{a.s.}, \quad \text{as } n \rightarrow \infty.$$

Now for  $u \leq u_1$ , we have  $\mathbb{P}_X(g_u) \geq \mathbb{P}_X(g_{u_1})$  and from (2.4),

$$\begin{aligned} -\delta_n &\leq \mathbb{P}_n(g_u) - \mathbb{P}_X(g_u) \leq \delta_n \\ -\delta_n + \mathbb{P}_X(g_u) &\leq \mathbb{P}_n(g_u) \leq \delta_n + \mathbb{P}_X(g_u), \end{aligned}$$

since  $\mathbb{P}_n(g_u) \geq \mathbb{P}_X(g_u) - \delta_n > 0$  for  $n$  large enough, then  $(\mathbb{P}_n(g_u))^{-1} \leq (\mathbb{P}_X(g_{u_1}) - \delta_n)^{-1}$ . We also have

$$(2.5) \quad |\mathbb{P}_X(f_u)| = \left| \int_u x dF(x) \right| \leq \left| \int_{\mathbb{R}} x dF(x) \right| \leq \int_{\mathbb{R}} |x| dF(x) = \mathbb{E}|X| = \alpha < \infty.$$

Thus

$$\sup_{u \leq u_1} |e_n(u) - e(u)| \leq \epsilon_n \left[ \mathbb{P}_X(g_{u_1}) - \delta_n \right]^{-1} + \alpha \left[ \mathbb{P}_X(g_{u_1}) (\mathbb{P}_X(g_{u_1}) - \delta_n) \right]^{-1} \delta_n$$

and then

$$\sup_{u \leq u_1} |e_n(u) - e(u)| \rightarrow 0 \quad \text{a.s.} \quad \text{as } n \rightarrow \infty. \quad \square$$

### 3. ASYMPTOTIC NORMALITY OF $e_n(u)$

In this section, we are concerned with weak laws of the empirical mean excess process as a stochastic process. Hereafter  $\{\mathbb{G}(g), g \in \mathcal{G}\}$  denotes a Gaussian centered functional stochastic process with variance-covariance function

$$\Gamma(g_1, g_2) = \int (g_1(x) - \mathbb{E}g_1(X_1))(g_2(x) - \mathbb{E}g_2(X_1))dF(x).$$

**Theorem 2.** *Let  $X_1, X_2, \dots$  be iid rv's with common finite second moment.*

*Put  $I = [u_0, u_1]$ , with  $u_0 < u_1 < x_F$  and define the functions of  $t \in \mathbb{R}$ ,*

$$h_u(t) = \mathbb{P}_X(g_u)^{-1} f_u(t) - \mathbb{P}_X(f_u) \mathbb{P}_X^{-2}(g_u) g_u(t) \quad \text{for } u \in I.$$

*Suppose that  $F$  is continuous and satisfies*

$$\limsup_{\delta \rightarrow 0} \sup_{(v, v-\delta) \in I^2} \left( \frac{F(v) - F(v-\delta)}{\sqrt{\delta}} \right)^2 = 0.$$

*Then the functional empirical processes  $\{\mathbb{G}_n(g_u), u \in I\}$  and  $\{\mathbb{G}_n(f_u), u \in I\}$  weakly converge respectively to  $\{\mathbb{G}(g_u), u \in I\}$  and  $\{\mathbb{G}(f_u), u \in I\}$  in  $\ell^\infty(I)$ .*

*And  $\{\sqrt{n}(e_n(u) - e(u)), u \in I\}$  weakly converges to  $\{\mathbb{G}(h_u), u \in I\}$ .*

Before we give the proof, we need this lemma.

**Lemma 1.** Let  $g$  be a finite measurable function defined on  $\mathbb{R}$  such that  $\mathbb{E}g(X_1)^2 < \infty$ . Let  $u_0 < u_1 < x_F$ . Define for any fixed  $v \in \mathbb{R}$  and  $\delta > 0$

$$\sigma^2(v, \delta) = \int_{v-\delta}^v (g(x) - \mathbb{E}(g(x)))^2 dF(x).$$

Let for a fixed  $n \geq 1$ ,  $u \in \mathbb{R}$

$$S_n(u) = \frac{1}{\sqrt{n}} \sum_{j=1}^n [g(X_j) \mathbb{I}_{(X_j > u)} - \mathbb{E}g(X_j) \mathbb{I}_{(X_j > u)}].$$

$$\text{If } \sup_{u_0 \leq v \leq u_1} \frac{\sigma^4(v, \delta)}{\delta} \rightarrow 0 \text{ as } \delta \rightarrow 0 \quad \text{and} \quad \sup_{u_0 \leq x \leq u_1} |g(x) - \mathbb{E}g(X)| < \infty,$$

then

$$\lim_{\delta \rightarrow 0} \sup_{u_0 \leq v \leq u_1} \sup_{n \geq 1} \frac{1}{\delta} P\left(\sup_{v-\delta \leq u \leq v} |S_n(u) - S_n(v)| \geq \eta\right) = 0.$$

**Proof of Lemma 1.** We fix  $v \in \mathbb{R}$  and consider  $\alpha = \sup_{v-\delta < u < v} |S_n(u) - S_n(v)|$ . Observe that for  $u < v$ ,

$$S_n(u) - S_n(v) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \{g(X_j) \mathbb{I}_{[u, v]}(X_j) - \mathbb{E}g(X_j) \mathbb{I}_{[u, v]}(X_j)\}.$$

Since for all  $(u, v) \in \mathbb{R}^2$ , we have

$$|S_n(v) - S_n(u)| \leq \frac{1}{\sqrt{n}} \sum_{j=1}^n [|g(X_j)| + |\mathbb{E}g(X_j)|] < \infty,$$

it comes that  $\alpha$  is finite. So for any  $\varepsilon > 0$ , we can find  $\bar{u} \in [v - \delta, v[$  such that,

$$(3.1) \quad |S_n(\bar{u}) - S_n(v)| \geq \alpha - \varepsilon.$$

Now, let  $\delta > 0$ . Define for any  $p \geq 1$ , and consider  $u_j(p) = u_j = v - \delta + j\delta/p$ ,  $j = 0, \dots, p$ .

Let us prove that for  $\varepsilon > 0$ ,

$$\lim_{p \rightarrow \infty} \max_{0 \leq j \leq p} |S_n(u_j) - S_n(v)| \geq \alpha - \varepsilon.$$

For each  $p \geq 1$ , let  $j$  such that

$$u_{j-1}(p) \leq \bar{u} \leq u_j(p).$$

We have,

$$\begin{aligned} |S_n(u_j) - S_n(v)| &\geq |S_n(\bar{u}) - S_n(v)| - \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i) \mathbb{I}_{[\bar{u}, u_j(p)]}(X_i) - \mathbb{E}g(X_i) \mathbb{I}_{[\bar{u}, u_j(p)]}(X_i)) \right| \\ &\geq |S_n(\bar{u}) - S_n(v)| - R_j(p), \end{aligned}$$

by denoting

$$R_j(p) = \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i) \mathbb{I}_{[\bar{u}, u_j(p)]}(X_i) - \mathbb{E}g(X_i) \mathbb{I}_{[\bar{u}, u_j(p)]}(X_i)) \right|.$$

We get from (3.1)

$$\max_{0 \leq j \leq p} |S_n(u_j) - S_n(v)| \geq \alpha - \varepsilon + R_j(p).$$

For a fixed  $n \geq 1$ ,  $R_j(p) \rightarrow 0$  as  $p \rightarrow \infty$ , since the sequence of intervals  $([\bar{u}, u_j(p)])_{p \geq 1}$  decreases to the empty set as  $p \rightarrow \infty$ .

Next, consider the collection points  $\{u_j(\ell), 0 \leq j \leq p, 1 \leq \ell \leq p\}$  and denote the set of its distinct values between them as  $\{\bar{u}_j, 1 \leq j \leq m(p)\}$ . We still have  $|\bar{u}_j - \bar{u}_{j-1}| \leq \delta/p$ . And we surely have for any  $\varepsilon > 0$

$$\lim_{p \rightarrow \infty} \max_{0 \leq j \leq m(p)} |S_n(\bar{u}_j) - S_n(v)| \geq \alpha - \varepsilon$$

and then

$$\lim_{p \rightarrow \infty} \max_{0 \leq j \leq m(p)} |S_n(\bar{u}_j) - S_n(v)| \geq \alpha$$

and finally

$$\sup_{p \geq 1} \max_{0 \leq j \leq m(p)} |S_n(\bar{u}_j) - S_n(v)| = \alpha.$$

By construction,  $\max_{0 \leq j \leq m(p)} |S_n(\bar{u}_j) - S_n(v)|$  is non decreasing in  $p$ . So, by the Monotone Convergence Theorem, for any fixed  $v > 0$ , for any  $\eta > 0$ ,

$$(3.2) \quad \mathbb{P}\left(\sup_{v-\delta \leq u \leq v} |S_n(u) - S_n(v)| \geq \eta\right) = \lim_{p \uparrow \infty} \mathbb{P}\left(\max_{1 \leq j \leq m(p)} |S_n(\bar{u}_j) - S_n(v)| \geq \eta\right).$$

$$\text{Put } Z_h = \sum_{i=1}^n \left( g(X_i) \mathbb{I}_{[\bar{u}_{h-1}, \bar{u}_h]}(X_i) - \mathbb{E}g(X_i) \mathbb{I}_{[\bar{u}_{h-1}, \bar{u}_h]}(X_i) \right), \quad h \geq 1.$$

We have

$$\sqrt{n}(S_n(\bar{u}_j) - S_n(v)) = \sum_{h=j}^{m(p)} Z_h = T_{m(p)-j}$$

with

$$\sqrt{n}(S_n(v - \delta) - S_n(v)) = \sum_{i=1}^{m(p)} Z_i = T_{m(p)} = T(n, u, \delta).$$

We observe that  $\{T_1, T_2, \dots, T_{m(p)}\}$  are partial sums of i.i.d. centered random variables so that the  $T_j^4$  form a submartingale. By the maximal inequality form submartingales, for any fixed  $p$

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq j \leq m(p)} |S_n(\bar{u}_j) - S_n(v)| \geq \eta\right) &= \mathbb{P}\left(\max_{1 \leq j \leq m(p)} |T_j| \geq \eta\sqrt{n}\right) \leq \frac{1}{\eta^4 n^2} \mathbb{E}T_{m(p)}^4 \\ &\leq \frac{1}{\eta^4 n^2} \mathbb{E}T(n, u, \delta)^4. \end{aligned}$$

Since the right hand does not depend on  $p$ , we get by (3.2)

$$\frac{1}{\delta} \mathbb{P}\left(\sup_{v-\delta \leq u \leq v} |S_n(u) - S_n(v)| \geq \eta\right) \leq \frac{1}{\delta \eta^4 n^2} \mathbb{E}T(n, u, \delta)^4.$$

Notice that  $T(n, u, \delta)$  is a sum of  $n$  i.i.d centered random variables with variance

$$\kappa_1(v, \delta) = \sigma^2(v, \delta) = \int_{v-\delta}^v (g(x) - \mathbb{E}(g(x)))^2 dF(x)$$

and fourth moment

$$\kappa_2(v, \delta) = \int_{v-\delta}^v (g(x) - \mathbb{E}(g(x)))^4 dF(x).$$

Simple computations give (see the APPENDIX 7.1 for a simple proof of that)

$$\mathbb{E}(T(n, u, \delta))^4 = n\kappa_2(v, \delta) + 3n(n-1)\kappa_1^2(v, \delta).$$

By putting these facts together, we arrive at

$$\begin{aligned} \frac{1}{\delta} \mathbb{P} \left( \sup_{v-\delta \leq u \leq v} |S_n(u) - S_n(v)| \geq \eta \right) &\leq \eta^{-4} \left( \frac{n\kappa_2(v, \delta) + 3n(n-1)\sigma^4(v, \delta)}{\delta n^2} \right) \\ &\leq \eta^{-4} \left( \frac{\kappa_2(v, \delta)}{\delta} \times \frac{1}{n} + 3 \frac{\sigma^4(v, \delta)}{\delta} \times \left[ 1 - \frac{1}{n} \right] \right). \end{aligned}$$

Remark that

$$\begin{aligned} \sup_{u_0 \leq v \leq u_1} \frac{\kappa_2(v, \delta)}{\delta} &\leq \left( \sup_{u_0 \leq x \leq u_1} |g(x) - \mathbb{E}g(X)| \right)^4 \times \delta^{-1} \times \sup_{u_0 \leq v \leq u_1} \int_{v-\delta}^v dF(x) \\ &\leq \left( \sup_{u_0 \leq x \leq u_1} |g(x) - \mathbb{E}g(X)| \right)^4 \times \sup_{u_0 \leq v \leq u_1} \frac{F(v) - F(v-\delta)}{\delta}. \end{aligned}$$

We finally get

$$\lim_{\delta \rightarrow 0} \sup_{u_0 \leq v \leq u_1} \sup_{n \geq 1} \frac{1}{\delta} \mathbb{P} \left( \sup_{v-\delta \leq u \leq v} |S_n(u) - S_n(v)| \geq \eta \right) = 0$$

$$\text{whenever } \lim_{\delta \rightarrow 0} \sup_{u_0 \leq v \leq u_1} \frac{\sigma^4(v, \delta)}{\delta} = 0 \text{ and } \sup_{u_0 \leq x \leq u_1} |g(x) - \mathbb{E}g(X)| < +\infty.$$

□

This achieves the proof of the lemma.

### Proof of Theorem 2.

By Theorem 2.7.5 in [10] applied to  $\mathcal{F}_1$  and by the fact that  $\mathcal{F}_2$  is a Vapnik-Chervonenkis class, condition (2.5.1) is satisfied for both  $\mathcal{F}_1$  and  $\mathcal{F}_2$  thus  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are Donsker classes.

This may be used in a simple manner to get

$$(3.3) \quad A_n = \max(\sup_{u \in I} |\mathbb{G}_n(g_u)|, \sup_{u \in I} |\mathbb{G}_n(f_u)|) = O_{\mathbb{P}}(1, I) \text{ as } n \rightarrow \infty.$$

Denote the functional empirical process for any real function  $g$  by

$$\mathbb{G}_n(g) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \{g(X_i) - \mathbb{E}g(X_i)\}.$$

Remind that for any Donsker class  $\mathcal{G}$ , the functional stochastic process  $\{\mathbb{G}_n(g), g \in \mathcal{G}\}$  converges in law to a Gaussian and centered stochastic process  $\{\mathbb{G}(g), g \in \mathcal{G}\}$  whose variance-covariance function is

$$\Gamma(g_1, g_2) = \int (g_1(x) - \mathbb{E}g(X_1))(g_2(x) - \mathbb{E}g_2(X_1))dF(x).$$

We have, as  $n \rightarrow \infty$

$$\mathbb{P}_n(g_u) = \mathbb{P}_X(g_u) + \frac{\mathbb{G}_n(g_u)}{\sqrt{n}}$$

$$\mathbb{P}_n(f_u) = \mathbb{P}_X(f_u) + \frac{\mathbb{G}_n(f_u)}{\sqrt{n}}.$$

Thus

$$\begin{aligned}
\sqrt{n}(e_n(u) - e(u)) &= \sqrt{n} \left( \frac{\mathbb{P}_n(f_u)}{\mathbb{P}_n(g_u)} - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} \right) \\
&= \sqrt{n} \left( \frac{\mathbb{P}_n(f_u)}{\mathbb{P}_n(g_u)} - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_n(g_u)} + \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_n(g_u)} - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} \right) \\
&= \frac{1}{\mathbb{P}_n(g_u)} \sqrt{n} \left( \mathbb{P}_n(f_u) - \mathbb{P}_X(f_u) \right) - \mathbb{P}_X(f_u) \frac{\sqrt{n} \left( \mathbb{P}_n(g_u) - \mathbb{P}_X(g_u) \right)}{\mathbb{P}_n(g_u) \mathbb{P}_X(g_u)} \\
&= \frac{1}{\mathbb{P}_n(g_u)} \left[ \mathbb{G}_n(f_u) - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} \mathbb{G}_n(g_u) \right] \\
&= \frac{1}{\mathbb{P}_n(g_u)} \left[ \mathbb{G}_n \left( f_u - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} g_u \right) \right].
\end{aligned}$$

We find

$$\begin{aligned}
(\mathbb{P}_n(g_u))^{-1} &= \left[ \mathbb{P}_X(g_u) + \frac{\mathbb{G}_n(g_u)}{\sqrt{n}} \right]^{-1} \\
&= \mathbb{P}_X^{-1}(g_u) \left[ 1 + \mathbb{P}_X^{-1}(g_u) \times n^{-1/2} \times \mathbb{G}_n(g_u) \right]^{-1} \\
&= \mathbb{P}_X^{-1}(g_u) \left[ 1 - \mathbb{P}_X^{-1}(g_u) \times n^{-1/2} \times \mathbb{G}_n(g_u) + \mathbb{P}_X^{-1}(g_u) \times \theta \left( n^{-1/2} \times \mathbb{G}_n(g_u) \right) \right]
\end{aligned}$$

Since  $\mathcal{F}_1$  is a Donsker class, then  $\sup_{u \in I} |\mathbb{G}_n(g_u)| = \|\mathbb{G}_n\|_{\mathcal{F}_1} = O_{\mathbb{P}}(1, I)$ . So

$$(\mathbb{P}_n(g_u))^{-1} = \mathbb{P}_X^{-1}(g_u) \left[ 1 - \mathbb{P}_X^{-1}(g_u) \times n^{-1/2} \times O_{\mathbb{P}}(1, I) \right]$$

Let us remind that  $h_u = \mathbb{P}_X(g_u)^{-1} f_u - \mathbb{P}_X(f_u) \mathbb{P}_X^{-2}(g_u) g_u$ . Then, for  $u \in I$ , we get

$$\begin{aligned}
\sqrt{n}(e_n(u) - e(u)) &= \left[ \mathbb{G}_n \left( f_u - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} g_u \right) \right] \times \left[ \mathbb{P}_X^{-1}(g_u) - \mathbb{P}_X^{-2}(g_u) \times n^{-1/2} \times O_{\mathbb{P}}(1, I) \right] \\
&= \mathbb{G}_n(h_u) + \mathbb{G}_n(h_u) \times \mathbb{P}_X^{-1}(g_u) \times n^{-1/2} \times O_{\mathbb{P}}(1, I).
\end{aligned}$$

We finally have

$$(3.4) \quad \sqrt{n}(e_n(u) - e(u)) = \mathbb{G}_n(h_u) + \mathbb{G}_n(h_u) \times o_{\mathbb{P}}(1, I). \quad \square$$

**Lemma 2.** *The class  $\mathcal{F}_3 = \left\{ h_u = \frac{f_u}{\mathbb{P}_X(g_u)} - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} g_u, u \in I \right\}$  is a Donsker Class.*

At this step, we want to prove that  $\mathcal{F}_3 = \{h_u, u_0 \leq u \leq u_1\}$  is a Donsker Class. Since we obviously have, by the *Central Limit Theorem*, finite distribution convergence of  $\{\mathbb{G}_n(h_u), u \in I\}$  to the stochastic process  $\{\mathbb{G}(h_u), u \in I\}$  in  $\ell^\infty(\mathcal{F}_3)$ , we only need to prove the asymptotic tightness of  $\{\mathbb{G}_n(h_u), u \in I\}$ .

In view of Theorem in 1.5.7 in [10], it is enough to prove that

$$\lim_{\delta \rightarrow 0} \sup_{u \in I} \lim_{n \rightarrow \infty} \sup_{\delta} \frac{1}{\delta} \mathbb{P} \left( \sup_{v - \delta \leq u \leq v} |\mathbb{G}_n(h_u) - \mathbb{G}_n(h_v)| \geq \eta \right) = 0.$$

Here, we apply **Lemma 1** for the nondecreasing measurable function  $g(x) = x$  and  $g(x) = 1$ .

In both cases, we inspect the assumptions of this lemma and see that if  $g(x) = x$ ,



we get  $g(x) \leq g(u_1) = u_1$  for any  $u_0 \leq x \leq u_1$  and thus

$$\begin{aligned} \sup_{u_0 \leq v \leq u_1} \frac{\sigma^4(v, \delta)}{\delta} &= \sup_{u_0 \leq v \leq u_1} \frac{1}{\delta} \left( \int_{v-\delta}^v (g(x) - \mathbb{E}g(x))^2 dF(x) \right)^2 \\ &\leq |u_1 - \mathbb{E}(X)|^4 \times \sup_{u_0 \leq v \leq u_1} \left( \frac{F(v) - F(v-\delta)}{\sqrt{\delta}} \right)^2 \rightarrow 0 \text{ as } \delta \rightarrow 0, \end{aligned}$$

and

$$\sup_{x \in I} |g(x) - \mathbb{E}g(X)| \leq |u_1| + |\mathbb{E}(X)| < \infty.$$

If  $g(x) = 1$ , the result is obvious.

We can apply **Lemma 1** and we will get,

$$\lim_{\delta \rightarrow 0} \sup_{u \in I} \limsup_{n \rightarrow \infty} \frac{1}{\delta} \mathbb{P} \left( \sup_{v-\delta \leq u \leq v} |\mathbb{G}_n(f_u) - \mathbb{G}_n(f_v)| \geq \eta \right) = 0$$

and

$$\lim_{\delta \rightarrow 0} \sup_{u \in I} \limsup_{n \rightarrow \infty} \frac{1}{\delta} \mathbb{P} \left( \sup_{v-\delta \leq u \leq v} |\mathbb{G}_n(g_u) - \mathbb{G}_n(g_v)| \geq \eta \right) = 0.$$

But by Theorem 8.3 of Billingsley [2], p.56, and by Theorem 2.2 in Lo [7], these two previous equalities entail, that

$$\lim_{\delta \rightarrow 0} \sup_{u \in I} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{G}_n(f_u) - \mathbb{G}_n(f_v)| \geq \eta \right) = 0$$

and

$$\lim_{\delta \rightarrow 0} \sup_{u \in I} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{G}_n(g_u) - \mathbb{G}_n(g_v)| \geq \eta \right) = 0.$$

Next, we use the following development for  $(u, v) \in I^2$

$$\begin{aligned} h_u - h_v &= \mathbb{P}_X^{-1}(g_u) \left( f_u - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X(g_u)} g_u \right) - \mathbb{P}_X^{-1}(g_v) \left( f_v - \frac{\mathbb{P}_X(f_v)}{\mathbb{P}_X(g_v)} g_v \right) \\ &= \underbrace{\mathbb{P}_X^{-1}(g_u) f_u - \mathbb{P}_X^{-1}(g_v) f_v}_{a(u,v)} - \underbrace{\left( \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} g_u - \frac{\mathbb{P}_X(f_v)}{\mathbb{P}_X^2(g_v)} g_v \right)}_{b(u,v)} \\ &= a(u,v) - b(u,v) \end{aligned}$$

We get

$$\begin{aligned} a(u,v) &= \mathbb{P}_X^{-1}(g_u) f_u - \mathbb{P}_X^{-1}(g_u) f_v + \mathbb{P}_X^{-1}(g_u) f_v - \mathbb{P}_X^{-1}(g_v) f_v \\ &= (f_u - f_v) \times \mathbb{P}_X^{-1}(g_u) + \left( \mathbb{P}_X^{-1}(g_u) - \mathbb{P}_X^{-1}(g_v) \right) \times f_v \\ &= \frac{f_u - f_v}{\mathbb{P}_X(g_u)} - \frac{\mathbb{P}_X(g_u) - \mathbb{P}_X(g_v)}{\mathbb{P}_X(g_u) \times \mathbb{P}_X(g_v)} \times f_v. \end{aligned}$$

Then

$$|\mathbb{G}_n(a(u,v))| \leq \frac{1}{\mathbb{P}_X(g_u)} \times |\mathbb{G}_n(f_u - f_v)| + \frac{|\mathbb{P}_X(g_u) - \mathbb{P}_X(g_v)|}{\mathbb{P}_X(g_u) \times \mathbb{P}_X(g_v)} \times \mathbb{G}_n(f_v).$$

Next

$$\begin{aligned} b(u,v) &= \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} \times g_u - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} \times g_v + \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} \times g_v - \frac{\mathbb{P}_X(f_v)}{\mathbb{P}_X^2(g_v)} \times g_v \\ &= (g_u - g_v) \times \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} + \left[ \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} - \frac{\mathbb{P}_X(f_v)}{\mathbb{P}_X^2(g_v)} \right] \times g_v. \end{aligned}$$

Next,

$$\begin{aligned}
\frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} - \frac{\mathbb{P}_X(f_v)}{\mathbb{P}_X^2(g_v)} &= \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_u)} - \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_v)} + \frac{\mathbb{P}_X(f_u)}{\mathbb{P}_X^2(g_v)} - \frac{\mathbb{P}_X(f_v)}{\mathbb{P}_X^2(g_v)} \\
&= \left( \frac{1}{\mathbb{P}_X^2(g_u)} - \frac{1}{\mathbb{P}_X^2(g_v)} \right) \times \mathbb{P}_X(f_u) \\
&\quad + \left( \mathbb{P}_X(f_u) - \mathbb{P}_X(f_v) \right) \times \frac{1}{\mathbb{P}_X^2(g_v)} \\
&= \left( \frac{(\mathbb{P}_X(g_v) + \mathbb{P}_X(g_u)) \times (\mathbb{P}_X(g_v) - \mathbb{P}_X(g_u))}{\mathbb{P}_X^2(g_u) \times \mathbb{P}_X^2(g_v)} \right) \times \mathbb{P}_X(f_u) \\
&\quad + \left( \mathbb{P}_X(f_u) - \mathbb{P}_X(f_v) \right) \times \frac{1}{\mathbb{P}_X^2(g_v)}.
\end{aligned}$$

Also,

$$\begin{aligned}
|\mathbb{G}_n(b(u, v))| &\leq \frac{|\mathbb{P}_X(f_u)|}{\mathbb{P}_X^2(g_u)} \times |\mathbb{G}_n(g_u - g_v)| \\
&\quad + \frac{|\mathbb{P}_X(g_v) + \mathbb{P}_X(g_u)| \times |\mathbb{P}_X(g_v) - \mathbb{P}_X(g_u)|}{\mathbb{P}_X^2(g_u) \times \mathbb{P}_X^2(g_v)} \times |\mathbb{P}_X(f_u)| \times |\mathbb{G}_n(g_v)| \\
&\quad + |\mathbb{P}_X(f_u) - \mathbb{P}_X(f_v)| \times |\mathbb{G}_n(g_v)| \times \frac{1}{\mathbb{P}_X^2(g_v)}.
\end{aligned}$$

For  $(u, v) \in [u_0, u_1]^2$ , let us use the bounds of  $\mathbb{P}_X^{-1}(g_u)$ ,  $\mathbb{P}_X^{-1}(g_v)$ , and  $\mathbb{P}_X(f_u)$ .

We obtain  $\mathbb{P}_X^{-1}(g_u) \leq (\bar{F}(u_1))^{-1}$ ,  $\mathbb{P}_X^{-1}(g_v) \leq (\bar{F}(u_1))^{-1}$ , and finally, from (2.5), we get  $|\mathbb{P}_X(f_u)| \leq \mathbb{E}|X|$ .

Thus by using these bounds and (3.3), it comes that

$$\begin{aligned}
\sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{G}_n(h_u - h_v)| &\leq B_1 \times \sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{G}_n(f_u - f_v)| + B_2 \times \sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{G}_n(g_u - g_v)| \\
&\quad + \left( B_3 \times \sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{P}_X(g_u) - \mathbb{P}_X(g_v)| + B_4 \times \sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{P}_X(f_u) - \mathbb{P}_X(f_v)| \right) A_n,
\end{aligned}$$

where

$$\begin{cases} B_1 = (\bar{F}(u_1))^{-1}; \\ B_2 = \mathbb{E}|X| \times (\bar{F}(u_1))^{-2}; \\ B_3 = \bar{F}(u_1)^{-2} \left( (\bar{F}(u_1))^{-2} + 2\bar{F}(u_0) \right); \\ B_4 = (\bar{F}(u_1))^{-2}; \\ A_n = \max(\sup_{u \in I} |\mathbb{G}_n(g_u)|, \sup_{u \in I} |\mathbb{G}_n(f_u)|). \end{cases}$$

Now we observe that

$$\sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{P}_X(g_u) - \mathbb{P}_X(g_v)| = \sup_{|u-v| \leq \delta, (u,v) \in I^2} |F(u) - F(v)|$$

and

$$\begin{aligned}
\sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{P}_X(f_u) - \mathbb{P}_X(f_v)| &\leq \sup_{|u-v| \leq \delta, (u,v) \in I^2} \left| \int_u^\infty t dF(t) - \int_v^\infty t dF(t) \right| \\
&\leq \sup_{|u-v| \leq \delta, (u,v) \in I^2} \left| \int_u^v t dF(t) \right| \\
&\leq \max(|u_0|, |u_1|) \sup_{|u-v| \leq \delta, (u,v) \in I^2} |F(u) - F(v)|.
\end{aligned}$$

These quantities go to zero whenever  $F$  is continuous and hence uniformly continuous in  $I$ . Putting all these facts together and using (3.3) yield

$$\sup_{n \geq 1} \sup_{|u-v| \leq \delta, (u,v) \in I^2} |\mathbb{G}_n(h_u - h_v)| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Finally  $\mathcal{F}_3$  is a Donsker class, thus  $\sup_{u \in I} |\mathbb{G}_n(h_u)| = O_{\mathbb{P}}(1, I)$  and we get from (3.4) that

$$\sqrt{n}(e_n(u) - e(u)) = \mathbb{G}_n(h_u) + o_{\mathbb{P}}(1, I).$$

□

This completes the proof.

Now we are going to concentrate on consistency bands for the mean excess function.

#### 4. CONSISTENCY BANDS

Now, we may use the uniform bands of the functional empirical processes based on Talagrand's inequality (see [8]) and the new methods introduced by Mason and al. [9] to obtain consistency bands of the mean excess function as follows.

**Theorem 3.** *Let  $X_1, X_2, \dots$ , be i.i.d random variables with finite second moment. Put  $I = [u_0, u_1]$ , with  $-\infty < u_0 < u_1 < x_F$ .*

*We suppose that  $F$  is continuous and satisfies*

$$(4.1) \quad \limsup_{\delta \rightarrow 0} \sup_{(v, v-\delta) \in I^2} \left( \frac{F(v) - F(v-\delta)}{\sqrt{\delta}} \right)^2 = 0.$$

*Then for any  $\varepsilon > 0$ , there exists  $n_0$  such that for  $n \geq n_0$ ,*

$$\mathbb{P}\left(e_n(u) - \frac{E_n}{\sqrt{n}} < e(u) < e_n(u) + \frac{E_n}{\sqrt{n}}, u \in I\right) \geq 1 - \varepsilon,$$

*with*

$$(4.2) \quad E_n = \frac{1}{\overline{F}(u_1) - D_1/\sqrt{n}} \left( D_2 + \frac{D_1 \times \mathbb{E}|X|}{\overline{F}(u_1)} \right),$$

*and where*

$$\begin{cases} D_1 = 2AA_1\sqrt{\log 2} + A_1 \\ D_2 = A A_1 M_1 \sqrt{\log M_1} + A_1, \\ M_1 = \max(2, \max(|u_0|, |u_1|)) \end{cases}$$

*$A$  and  $A_1$  are universal constants.*

The proof of this theorem is rather technical so we postpone it in the APPENDIX SUBSECTION 7.2.1 where we also state the fundamental Talagrand's inequality.

**Remark:** The validity condition (4.1) is quite very weak and is satisfied by most of the continuous usual distribution functions. Indeed if  $F$  is absolutely continuous with respect to the Lebesgue measure with derivative function  $f$ , we get by using the *mean value theorem*,

$$\left( \frac{F(v) - F(v - \delta)}{\sqrt{\delta}} \right)^2 \leq \delta \times \sup_{x \in [v - \delta, v]} f^2(x).$$

But  $\sup_{x \in [v - \delta, v]} f^2(x) < \infty$  whenever  $f$  is continuous, by a simple argument from real analysis. This allows consistency bands for a huge number of absolutely continuous distribution functions. All the examples in the SECTION 5 are devoted to simulations satisfy (4.1) through this argument.

Now, we are going to focus on the applications of our results.

## 5. SIMULATIONS AND APPLICATIONS

**5.1. Introduction.** The Mean excess function can be used in two ways :

- First, it can be used to distinguish heavy tailed models distribution and those with light tailed distribution. An increasing mean excess function  $e(u)$  indicates a heavy-tailed distribution and a decreasing mean excess function  $e(u)$  indicates a light-tailed distribution. The exponential distribution has a constant mean excess function and is considered a medium-tailed distribution .

Then the plot of the mean excess function tends to infinity for heavy-tailed distributions, decreases to zero for light-tailed distributions and remains constant for an exponential distribution.

- Secondly, it can be used for tail estimation with the help of the generalized Pareto distribution which can model the tails of another distribution.

Let  $F_u(x)$ , the excess distribution over threshold  $u$ , defined by

$$F_u(x) = \mathbb{P}(X - u \leq x | X > u)$$

with  $0 \leq x < x_F - u$ , where  $x_F \leq \infty$  is the right endpoint of  $F$ .

By using Theorem 7.20 in [1] , a natural approximation of  $F_u$  is a generalized Pareto distribution  $\mathcal{GPD}(\xi, \beta)$  which mean excess function is given by

$$(5.1) \quad e(u) = \frac{\beta}{1 - \xi} + \frac{\xi}{1 - \xi} u, \quad \text{provided that } \xi < 1.$$

If the empirical mean excess function plot looks linear, we can fit a  $\mathcal{GPD}(\xi, \beta)$  model whose parameters can be estimated by means of linear least squares method : given data  $\{(u_1, y_1), \dots, (u_n, y_n)\}$ , where  $u_i = X_i$  and  $y_i = e_n(u_i)$ ,  $i = 1, \dots, n$ , we estimate the parameters  $\xi$  and  $\beta$  to be

$$\hat{\xi} = \frac{\hat{a}}{\hat{a} + 1} \quad \text{and} \quad \hat{\beta} = \frac{\hat{b}}{\hat{a} + 1},$$

where

$$\hat{a} = \frac{n \sum_{i=1}^n u_i y_i - \sum_{i=1}^n u_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n u_i^2 - \left( \sum_{i=1}^n u_i \right)^2} \quad \text{and} \quad \hat{b} = \bar{y} - \hat{a} \bar{u},$$

with  $\bar{u} = \frac{1}{n} \sum_{i=1}^n u_i$  and  $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$  are the sample means of the observations on  $u$  and  $y$ , respectively.

As far as we are concerned, our goal is to estimate the mean excess function by consistency bounds.

In the remainder of this section, we are backing on the empirical mean excess function (*emef* for short) to construct graphical tools goodness of fit test.

In the first step we are considering a large set of distributions for which we draw the average *emef*. That means that we fix a distribution function and consider  $n = 6000$  samples from it, each sample size is 4000. Next we compute the average of the  $n = 6000$  empirical mean functions.

The graphs of these average mean empirical functions would serve as stallions in the following sense: each other sample having an alike *emef* will suggest such an underlying distribution.

We will use, as a special guest, the generalized hyperbolic (*Gh* for short) family of distribution functions. Nowadays, this family is very important in financial modeling.

In a second step we will try to use the obtained graphs as stallions for real data.

In this paper, we focus on monthly returns and log-returns of Dow Jones data. We will see that these data strongly suggest *Gh* model.

This section, beyond financial data, shows how to use the *emef* for goodness of fit testing purposes. It opens a great verity of applications for different types of data.

**5.2. Usual distributions.** To assess the performance of our estimator, we present a simulation study. We draw simulated *emefs* for standard distributions and next for *Gh* family of distribution functions

**5.2.1. *Emef* for standard distributions.** We consider some simple models that are listed in the TABLE 1 below where the used parameters are specified and the *emef* figures corresponding to each choice are displayed.

Distributions	Parameters	Figures
GPD	$\xi = 0.25, \beta = 1$	Figure 1
	$\xi = -0.75, \beta = 1$	
Pareto Exponential	$\alpha = 7, \lambda = 3$	Figure 2
	$\lambda = 2$	
Weibull	$\beta = 1, \tau = 3.6$	Figure 3
	$\beta = 1.5, \tau = 0.2$	
Burr Gomberz	$\alpha = 0.5, \lambda = 0.5, \tau = 5$	Figure 4
	$\alpha = 1, \lambda = 0.5$	
Gamma Beta	$\alpha = 2, \beta = 0.001$	Figure 5
	$\lambda = 7, \beta = 2$	
Lognormal Normal	$\mu = 0, \sigma = 1$	Figure 6
	$\mu = 0, \sigma = 1$	
Laplace	$\mu = 0, \sigma = 1, \tau = 0.5$	Figure 7
<i>t</i> Student Cauchy	$\nu = 5, \mu = 1$	Figure 8
	$\mu = 0, \delta = 1$	

TABLE 1. The *emef* for standard distributions

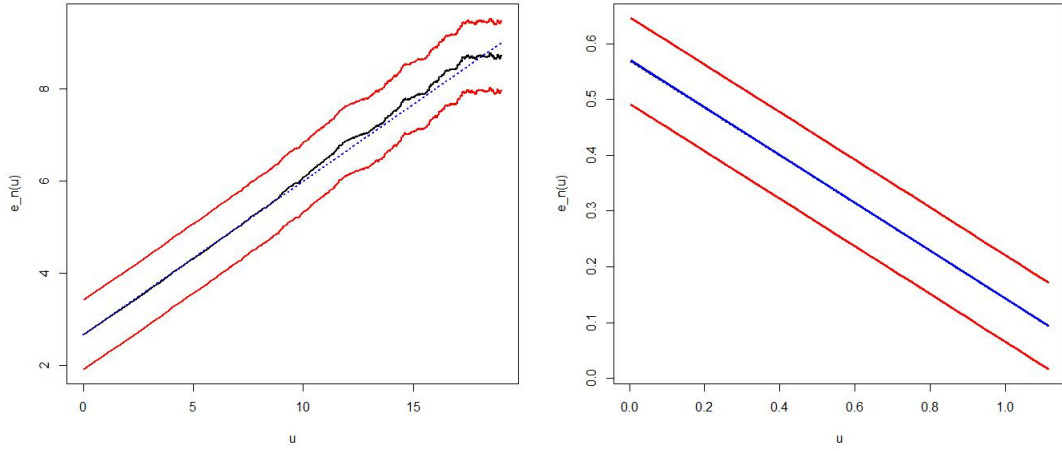


FIGURE 1. The *emef* for two generalized Pareto distributions : the left panel concerns the one with the parameters  $\xi = 0.25$ ,  $\beta = 1$  and the right panel concerns the one with the parameters  $\xi = -0.75$ ,  $\beta = 1$ .

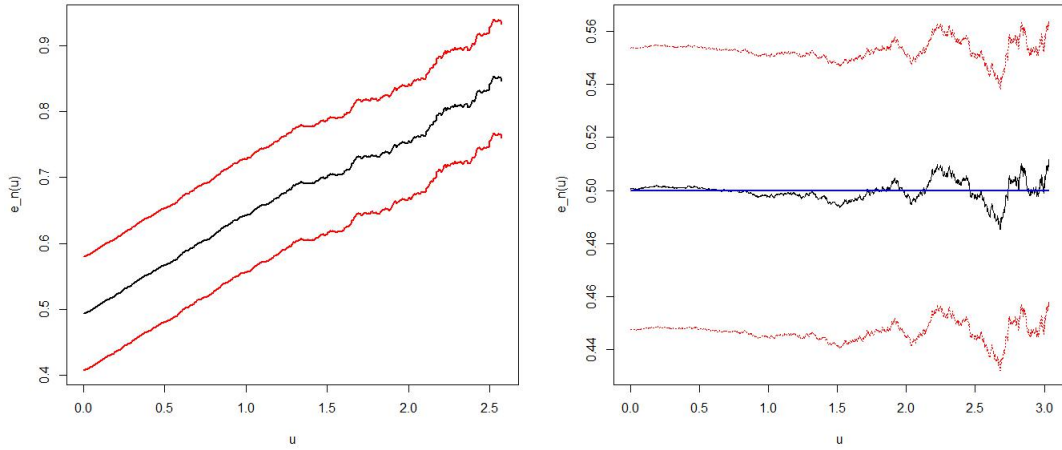


FIGURE 2. The left panel is the *emef* for a Pareto distribution with the parameters  $\alpha = 7$  and  $\lambda = 3$  and the right panel is the one for an Exponential distribution with the parameter  $\lambda = 2$ .

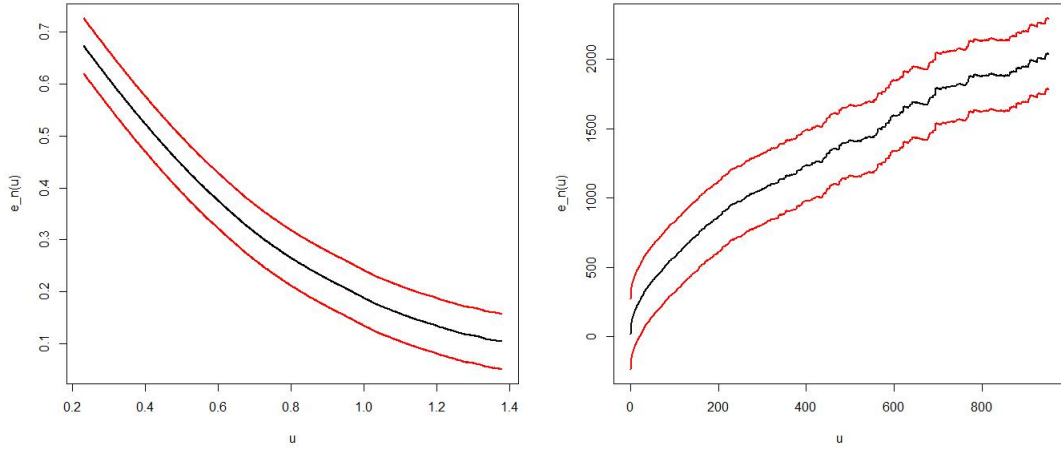


FIGURE 3. The *emef* for two Weibull distributions. The left panel concerns the one with the parameters  $\beta = 1$ ,  $\tau = 3.6$  and the right panel concerns the one with the parameters  $\beta = 1.5$ ,  $\tau = 0.2$ .

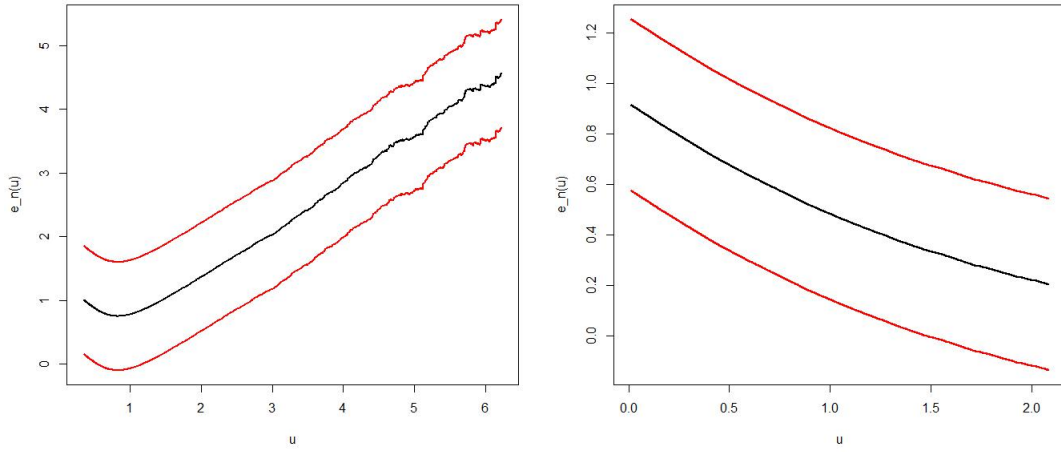


FIGURE 4. The left panel is the *emef* for the Burr distribution with the parameters  $\alpha = 0.5$ ,  $\lambda = 0.5$ ,  $\tau = 5$  and the right one is the *emef* for the Gompertz distribution with the parameters  $\alpha = 1$ ,  $\lambda = 0.5$ .

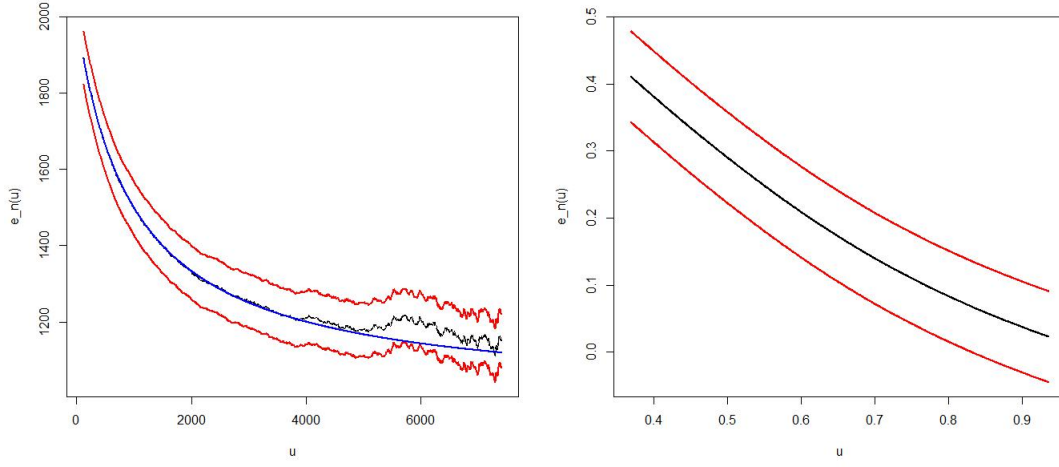


FIGURE 5. The left panel is the *emef* for the Gamma distribution with the parameters  $\alpha = 2$ ,  $\beta = 0.001$  and the right one is the *emef* for the Beta distribution with the parameters  $\lambda = 7$ ,  $\beta = 2$ .

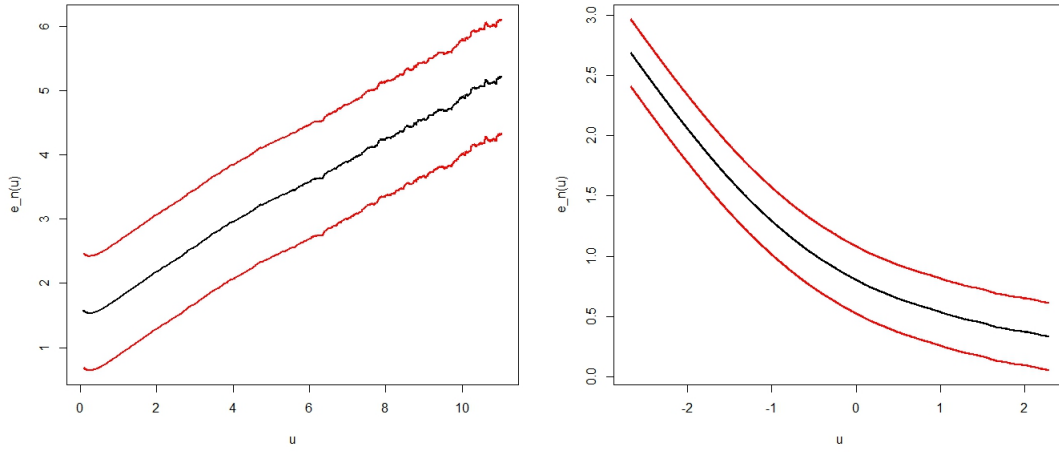


FIGURE 6. The left panel is the *emef* for the Lognormal distribution with mean  $\mu = 0$  and with variance  $\sigma^2 = 1$  and the right one is the *emef* for the Normal distribution with the same parameters.



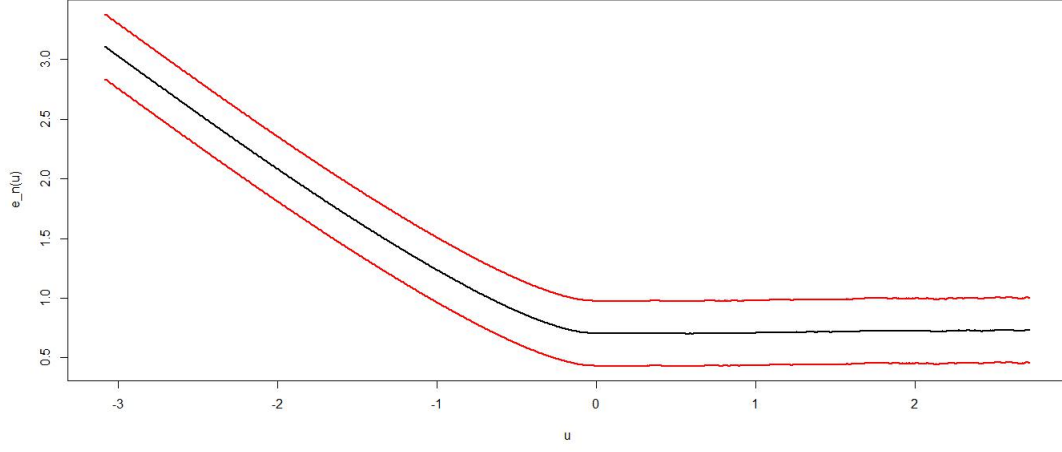


FIGURE 7. *Emef* for Laplace distribution with the parameters  $\mu = 0, \sigma = 1, \tau = 0.5$

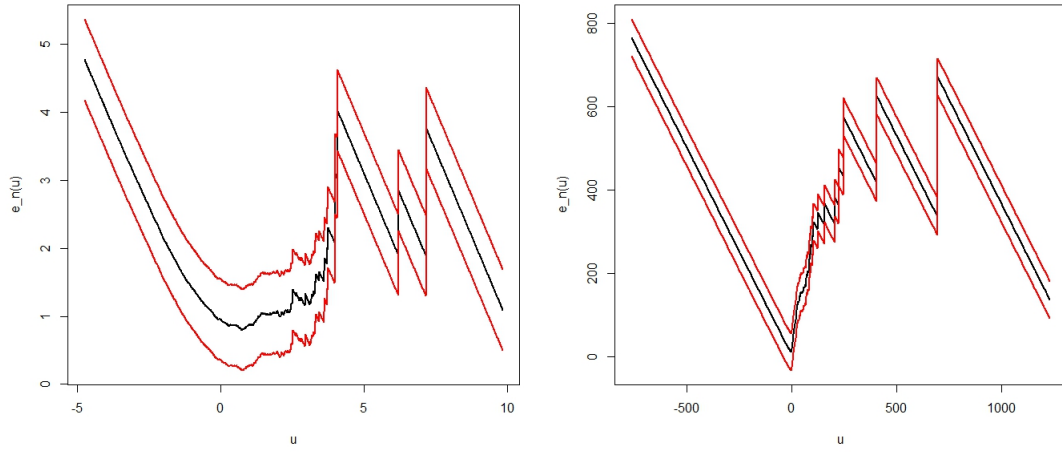


FIGURE 8. *Emef* for a *t* Student with  $\nu = 5$  degrees of freedom and skewing parameter  $\mu = 0$  (left panel) and for a Cauchy distribution with location parameter  $\mu = 0$  and scale parameter  $\delta = 1$  (right panel).

5.2.2. *Generalized hyperbolic models.* Next, we consider the *emefs* for the *Gh* models. We need some definitions. The Lebesgue density function of the one dimensional *Gh* is given by

$$(5.2) \quad f_{\lambda, \alpha, \beta, \delta, \mu}(x) = \mathbf{a}_{(\lambda, \alpha, \beta, \delta, \mu)} \times \left( \delta^2 + (x - \mu)^2 \right)^{(\lambda - \frac{1}{2})/2} e^{\beta(x - \mu)} \times K_{\lambda - \frac{1}{2}}(\alpha \sqrt{\delta^2 + (x - \mu)^2})$$

where

$$\mathbf{a}_{(\lambda, \alpha, \beta, \delta, \mu)} = \frac{(\alpha^2 - \beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi} \alpha^{(\lambda - \frac{1}{2})} \delta^\lambda K_\lambda(\delta \sqrt{\alpha^2 - \beta^2})}$$

is a norming constant to make the curve area equal to 1 and

$$K_\lambda(x) = \frac{1}{2} \int_0^\infty y^{\lambda-1} \exp\left(-\frac{1}{2}x(y + y^{-1})\right) dy, \quad (x > 0)$$

is the modified Bessel function of the third kind with index  $\lambda$ .

The dependence of the parameters  $\lambda, \alpha, \beta, \delta$ , and  $\mu$  is as follows:  $\alpha > 0$  determines the *shape*,  $0 \leq |\beta| < \alpha$  the *skewness*,  $\mu \in \mathbb{R}$  is a *location* parameter and  $\delta > 0$  serves for *scaling*. The parameter  $\lambda \in \mathbb{R}$  specifies the order  $K_\lambda$  function Bessel that appears in the *Gh* density function and is used to obtain different subclasses of *Gh* distribution.

In the following, we summarise the different domains of possibilities for the parameters

$$\begin{aligned} &\text{If } \lambda < 0, \quad (\delta > 0, \quad |\beta| \leq \alpha), \\ &\text{if } \lambda = 0, \quad (\delta > 0, \quad |\beta| < \alpha), \\ &\text{if } \lambda > 0, \quad (\delta \geq 0, \quad |\beta| < \alpha). \end{aligned}$$

An important *Gh* family aspect is that it embraces many special cases such that Hyperbolic ( $\lambda = 1$ ), Student-*t* ( $\lambda < 0$ ), Variance Gamma ( $\lambda > 0$ ), and the Normal Inverse Gaussian (NIG) ( $\lambda = -0.5$ ) distributions.

It nests also Generalized Inverse Gamma (GIG) distribution defined only by the three parameters  $\lambda, \alpha$ , and  $\beta$ . An Inverse Gaussian (IG) distribution is a GIG distribution with  $\lambda = -0.5$  and a Gamma ( $\Gamma$ ) distribution is also a GIG distribution with  $\beta = 0$ .

It contains some limiting distributions such as Cauchy distribution with parameters  $\mu$  and  $\delta$  (obtained for  $\lambda = -0.5$  and  $\alpha = \beta = 0$ ).

The Gaussian distribution with mean  $\mu$  and variance  $\sigma^2$  are obtained for  $\lambda = -0.5$ , for  $\alpha, \delta \rightarrow \infty$  and  $\frac{\delta}{\alpha} \rightarrow \sigma^2$ .

The Skew-Student *t* with  $\nu$  degrees of freedom is obtained if  $\alpha = |\beta|$ , then  $\nu = -2\lambda > 0$ .

The Student *t* distribution is obtained for  $\alpha = \beta = 0$ ,  $\mu = 0$  and  $\delta = \sqrt{\nu}$ . In the special case of hyperbolic distributions ( $\lambda = 1$ ), we obtain the skewed Laplace distribution for  $\delta = 0$ .

All of these have been used to model financial returns and log-returns.

In TABLE 2, we consider some specific  $Gh$  distributions with the superscript  $spe$  and limiting distributions with the superscript  $lm$ . The used parameters are specified and the  $emef$  figures corresponding to each choice are displayed.

Distributions	Parameters					Figures
	$\lambda$	$\alpha$	$\beta$	$\delta$	$\mu$	
Hyperbolique <sup>spe</sup>	1	1.5	-0.5	0.75	0.2	Figure 9
t-Stud. <sup>spe</sup>	-2	$10^{-8}$	0	2	0	
Asym. t-Stud. <sup>spe</sup>	-1.278	0.01186	0.01186	0.0766	1.005	Figure 10
	-1.247	0.0148	-0.0147	0.076	1.005	
NIG <sup>spe</sup>	-0.5	8.03	-1.37	0.051	0.0105	Figure 11
	-0.5	7.6	-1.24	0.052	0.0103	
Variance Gamma <sup>spe</sup>	2	0.3	0.1	2	0	Figure 12
GIG <sup>spe</sup>	5	3	1	-	-	
IG Inverse Gaussian <sup>lm</sup>	0.5	1	0	1	0	Figure 13
Gamma( $\alpha, \beta$ ) <sup>lm</sup>	0.5	$4.5 \cdot 10^{12}$	$10^{-8}$	-	-	
IT Inverse Gamma <sup>lm</sup>	-0.5	$1.9 \times 10^{-8}$	$3.1 \times 10^{-3}$	-	-	Figure 14
Skew Laplace <sup>lm</sup>	1	1.1	0.1	0.001	2	
Gaussian <sup>lm</sup> (3, 0.3)	-0.5	$10^6$	2	$3 \times 10^5$	3	Figure 15
Cauchy <sup>lm</sup> (7, 1)	-0.5	0	0	1	7	

TABLE 2. Specific and limiting  $GH$  distributions.

5.2.3. *Graphical test.* We are now in a position to use the  $emef$  graphs already drawn as tools of goodness of fit.

$Emef$  for Normal Inverse Gaussian (NIG) and  $t$ -student-distributions are not monotonic function. They decrease and increase like for  $emef$  returns data. For this reason, we fit them to both monthly returns and log-returns from Dow Jones data base (see FIGURE 17, FIGURE 19, FIGURE 21, and FIGURE 23).

Dow Jones data base consists of several compaignies like AXP(American Express compaigny), CSCO(Cisco Systems), DAX, CAT, IBM and so one. Each one having 5 values : from opening ( $op$ ) values to closing ( $cl$ ) values , also minimum ( $min$ ), maximum ( $max$ ), and volume ( $vol$ ) values.

We select AXP and CSCO compaignies and we consider returns and log-returns for their values as showed in the TABLE 3. Then we construct their  $emef$  plot and their fitted counterpart.

Estimates parameters and the  $emef$  are given in TABLE 4.

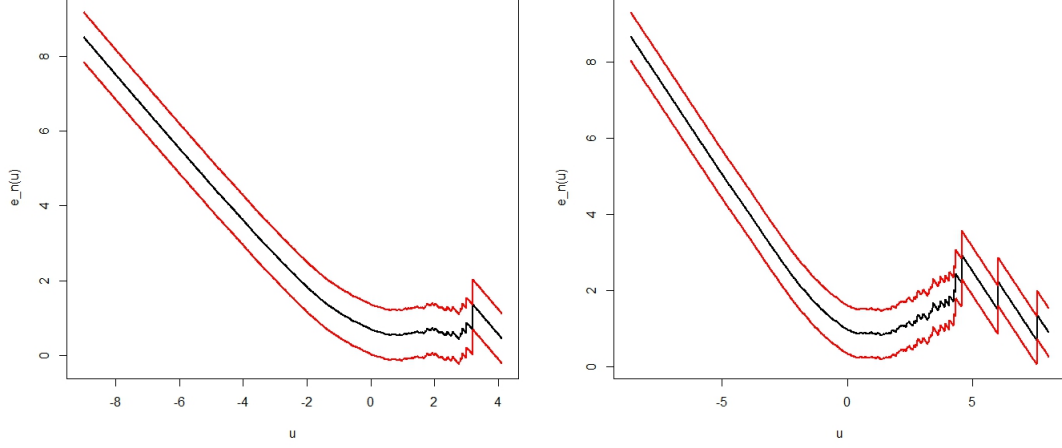


FIGURE 9. Left panel :  $Emef$  for an hyperbolic distribution with the parameters  $\lambda = 1$ ,  $\alpha = 1.5$ ,  $\beta = -0.5$ ,  $\delta = 0.75$ ,  $\mu = 0.2$ . Right panel :  $Emef$  for a  $t$ -student with the parameters  $\lambda = -2$ ,  $\alpha = 10^{-8}$ ,  $\beta = 0$ ,  $\delta = 2$ ,  $\mu = 0$ .

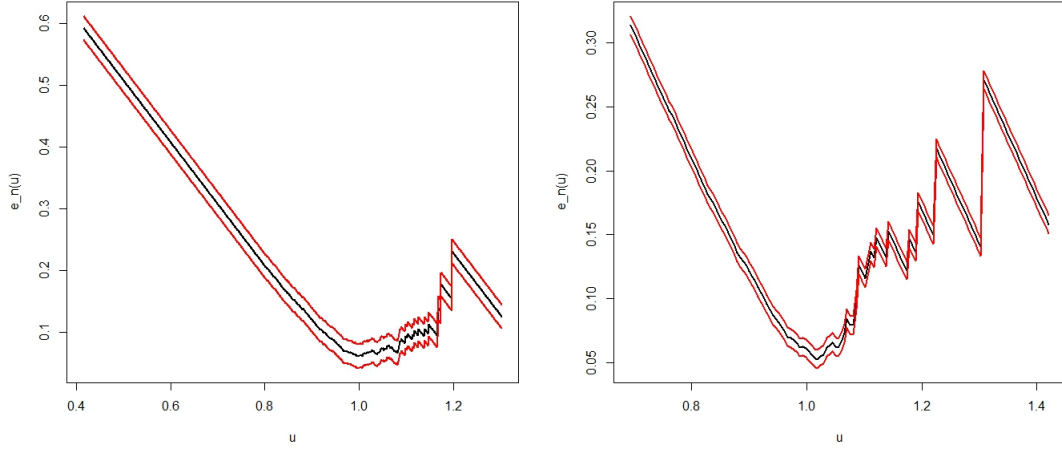


FIGURE 10. The  $emef$  for two  $t$ -Student distribution. The left panel concerns the one with the parameters  $\lambda = -1.278$ ,  $\alpha = 0.01186$ ,  $\beta = 0.01186$ ,  $\delta = 0.0766$ ,  $\mu = 1.005$  and the right panel concerns the one with the parameters  $\lambda = -1.247$ ,  $\alpha = 0.0148$ ,  $\beta = -0.0148$ ,  $\delta = 0.07683$ ,  $\mu = 1.005$ .

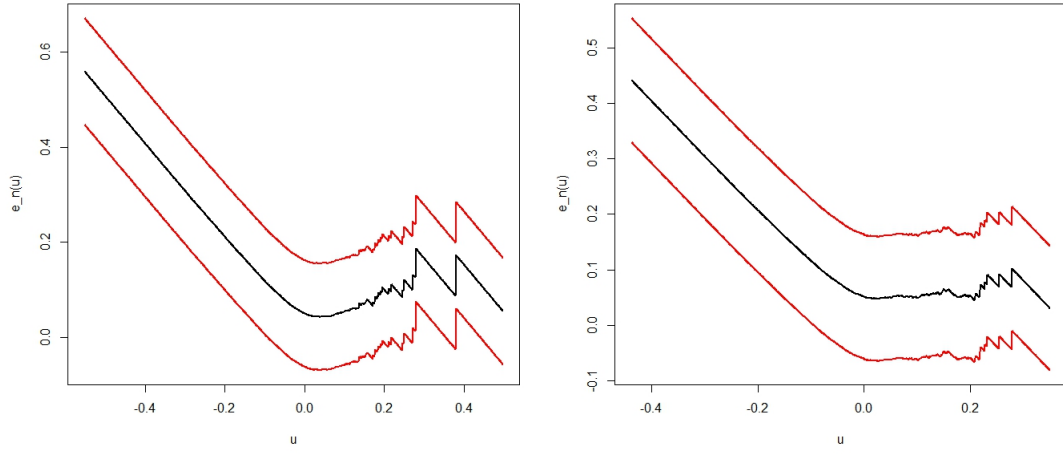


FIGURE 11. The *emef* for two Normal Inverse Gaussian distributions. The left panel concerns the one with the parameters  $\lambda = -0.5$ ,  $\alpha = 8.03$ ,  $\beta = -1.37$ ,  $\delta = 0.051$ ,  $\mu = 0.0105$  and the right panel concerns the one with the parameters  $\lambda = -0.5$ ,  $\alpha = 7.6$ ,  $\beta = -1.24$ ,  $\delta = 0.052$ ,  $\mu = 0.0103$ .

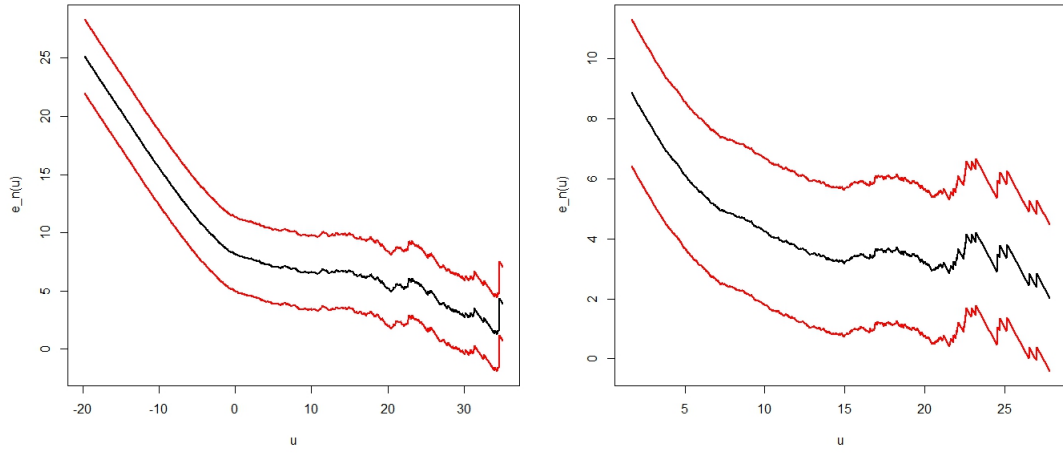


FIGURE 12. Left panel : the *emef* for a Variance Gamma distribution with the parameters  $\lambda = 2$ ,  $\alpha = 0.3$ ,  $\beta = 0.1$ ,  $\delta = 2$ ,  $\mu = 0$ . Right panel : the *emef* for a Generalized Inverse Gaussian distribution with the parameters  $\lambda = 5$ ,  $\alpha = 3$ ,  $\beta = 1$ .

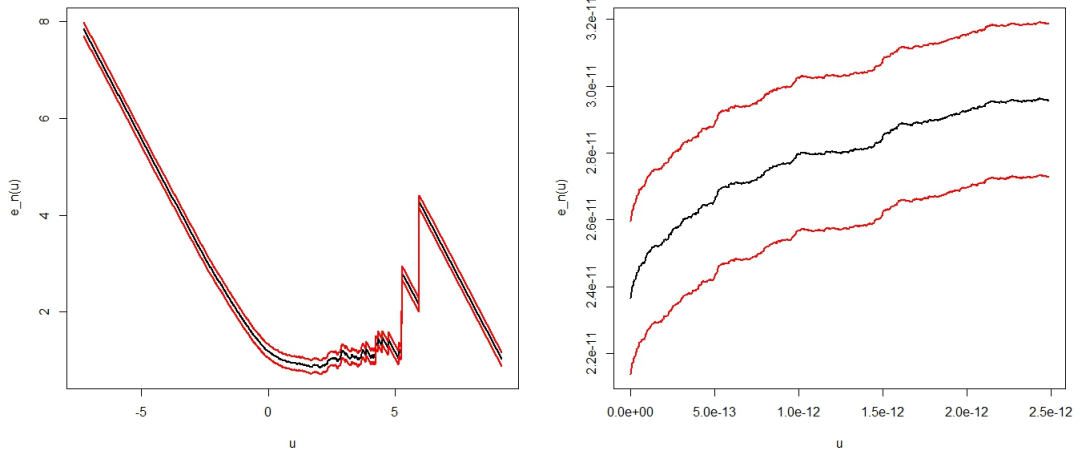


FIGURE 13. Left panel : the *emef* for Inverse Gaussian distribution with the parameters  $\lambda = 0.5$ ,  $\alpha = 1$ ,  $\beta = 0$ ,  $\delta = 1$ ,  $\mu = 0$ . Right panel : the *Emef* for a Gamma distribution with the parameters  $\lambda = 0.5$ ,  $\alpha = 4.5 \times 10^{12}$ ,  $\beta = 10^{-2}$ .

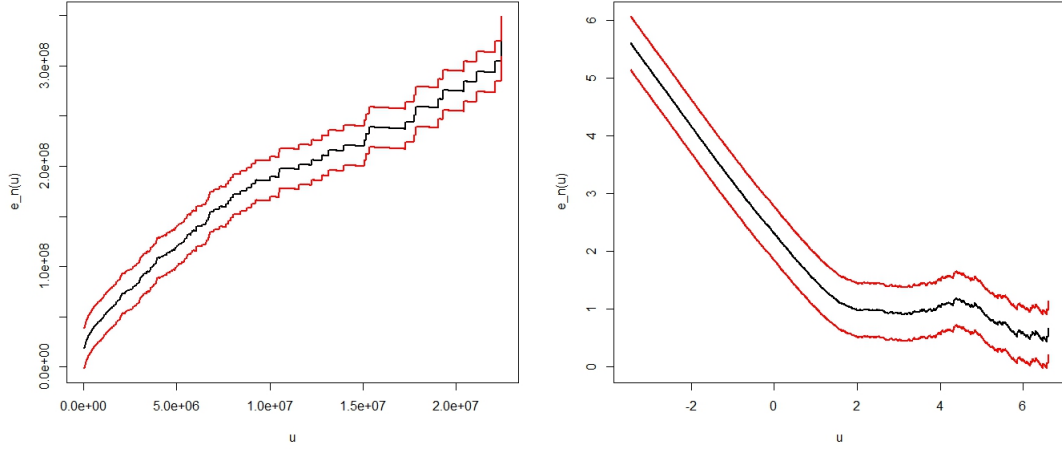


FIGURE 14. Left panel : the *emef* for the Inverse Gamma distribution with the parameters  $\lambda = -0.5$ ,  $\alpha = 1.9 \times 10^{-8}$ ,  $\beta = 3.1 \times 10^{-3}$ . Right panel : the *emef* for the Skew Laplace distribution with the parameters  $\lambda = 1$ ,  $\alpha = 1.1$ ,  $\beta = 0.1$ ,  $\delta = 10^{-3}$ ,  $\mu = 2$

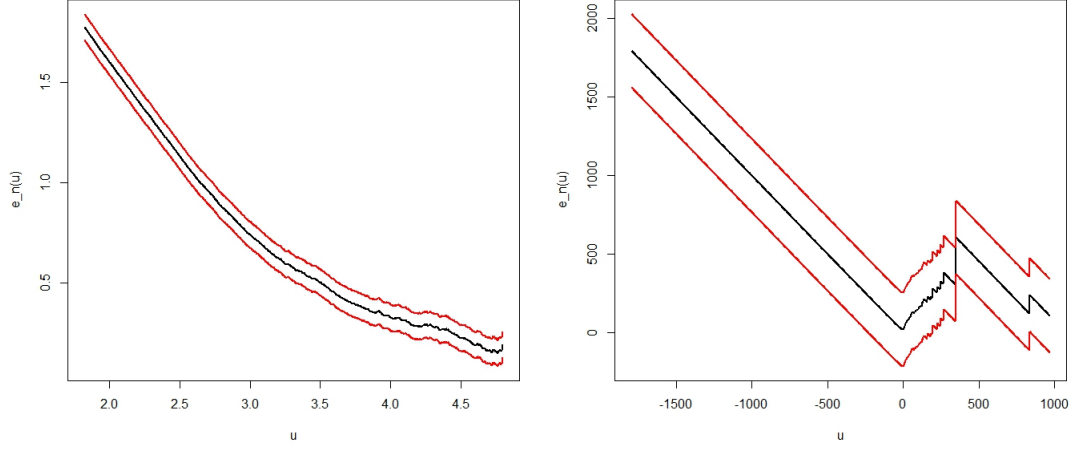


FIGURE 15. The left panel concerns the *emef* for the Gaussian distribution with mean  $\mu = 2$  and variance  $\sigma^2 = 0.3$ . The right panel concerns the *emef* for the Cauchy distribution with location parameter  $\mu = 7$  and scale parameter  $\delta = 1$ .

Compagnies	Nature	Values	<i>Emef</i> plots	
			<i>Real emef</i>	<i>Fitted emef</i>
AXP	Returns	op	Figure 16	Figure 17
		min		
	Log-returns.	max	Figure 18	Figure 19
		cl		
CSCO	Returns	min	Figure 20	Figure 21
		vol		
	Log-returns	op	Figure 22	Figure 23
		max		

TABLE 3. (Fitted) *Emef* for DAX and CSCO compagnies data.

Comp	Nature	Values	Ghyp estimates parameters					Fit.Dist	Figures
			$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\mu}$		
AXP	Returns	<i>op.</i>	-1.278	0.01186	0.0118	0.0766	1.005	<i>t</i> -stud	Fig. 17
		<i>min.</i>	-1.247	0.0148	-0.0148	0.0768	1.005	<i>t</i> -stud	
	Log-ret.	<i>max.</i>	-0.5	8.03	-1.37	0.051	0.0105	NIG	Fig. 19
		<i>cl.</i>	-0.5	7.6	-1.24	0.052	0.0103	NIG	
CSCO	Returns	<i>min.</i>	-1.24	0.0148	-0.0148	0.0768	1	<i>t</i> -Stud.	Fig. 21
		<i>vol.</i>	-3.82	4.22	4.22	0.613	0.753	<i>t</i> -Stud.	
	Log-ret.	<i>op.</i>	-1.26	0.83	- 0.83	0.07	0	<i>t</i> -Stud.	Fig. 23
		<i>max.</i>	-1.32	0.85	-0.85	0.076	0	<i>t</i> -Stud.	

TABLE 4. *Emef* for fitted *Gh* distributions to DAX and CSCO compagnies data.

5.2.4. *Commentaries.* In view of FIGURE 10 and FIGURE 17 we can say that *t* student distribution fits well opening and minimum values return for the American Express compaigny AXP, whereas *NIG* distribution fits well maximum and closing log-returns values for the Cysco System compaigny CSCO in view of FIGURE 11 and FIGURE 19.



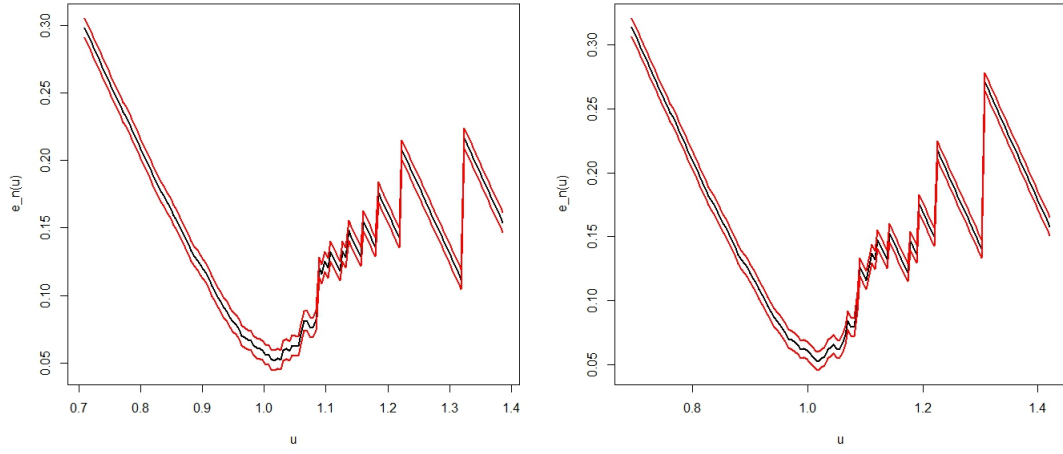


FIGURE 16. *Emef* for AXP compagny (monthly data returns). The left panel concerns the opening values and the right one concerns the minimum values.

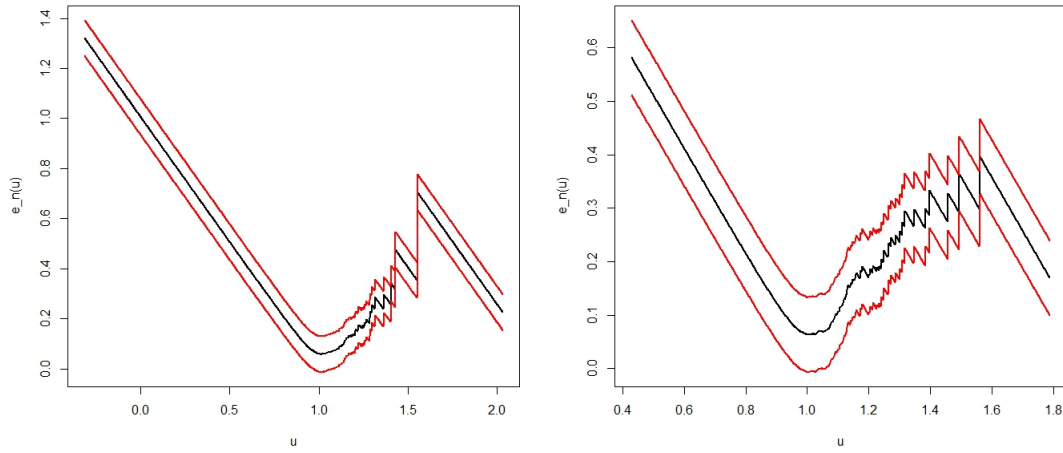


FIGURE 17. A *t*-Student distribution is fitted to monthly data returns for AXP compagny (see figure 16). The left panel concerns a *t* distribution with the parameters  $\lambda = -1.278$ ,  $\alpha = 0.01186$ ,  $\beta = 0.01186$ ,  $\delta = 0.0766$ , and  $\mu = 1.005$  fitted to opening values. The right one concerns a *t* distribution with the parameters  $\lambda = -1.247$ ,  $\alpha = 0.0148$ ,  $\beta = -0.0148$ ,  $\delta = 0.07683$ ,  $\mu = 1.005$  fitted to minimum values.

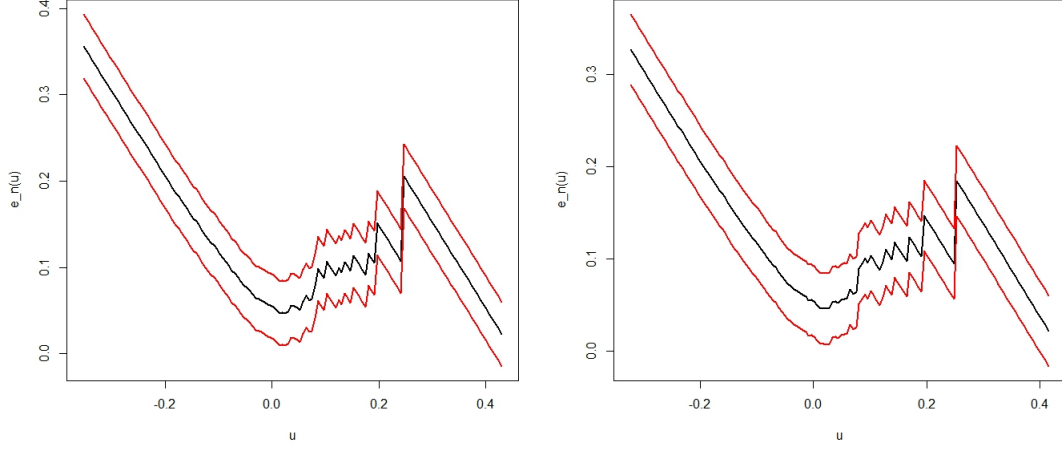


FIGURE 18.  $Emef$  for AXP compagny (monthly data log-returns). The left panel concerns the maximum values and the right panel concerns the closing values.

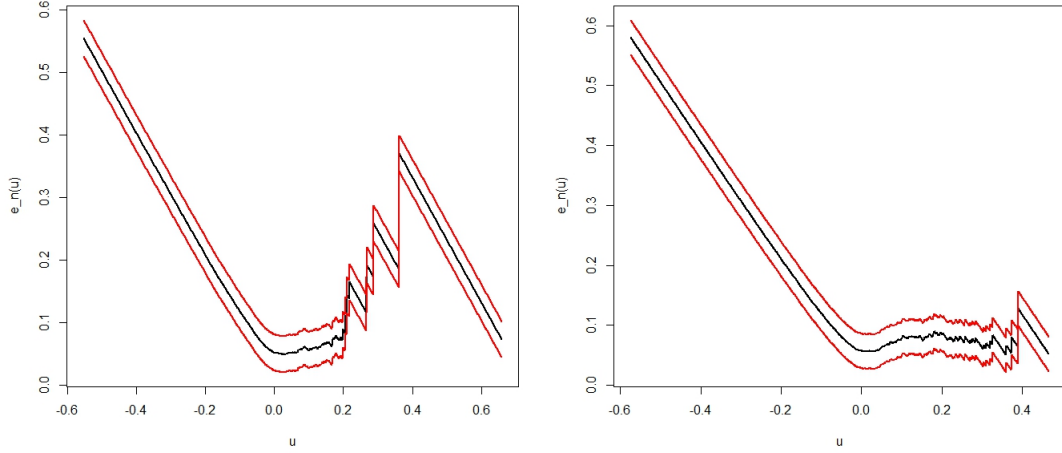


FIGURE 19. A NIG distribution is fitted to monthly data log-returns for AXP compagny (see Figure 18). The left panel concerns the one with the parameters  $\lambda = -0.5, \alpha = 8.03, \beta = -1.37, \delta = 0.051, \mu = 0.0105$  fitted to maximum values. The right panel concerns the one with the parameters  $\lambda = -0.5, \alpha = 7.6, \beta = -1.24, \delta = 0.052, \mu = 0.0103$  fitted to closing values.

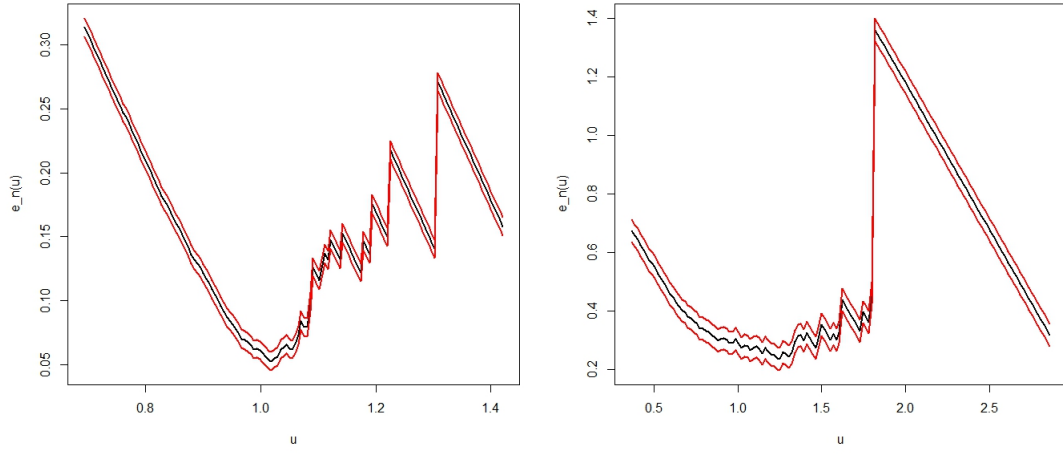


FIGURE 20.  $Emef$  for CSCO compagny (data returns). The left panel concerns monthly minimum values and the right panel concerns monthly volum values.

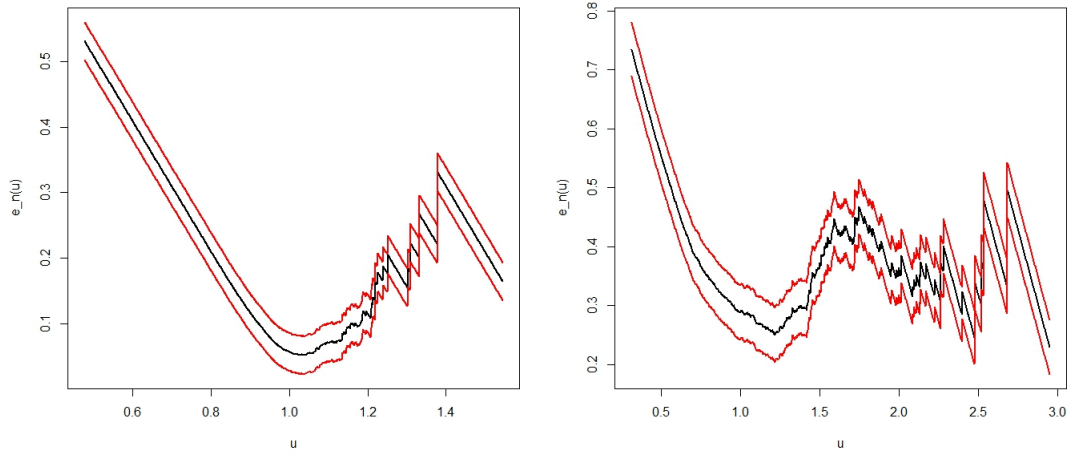


FIGURE 21. A  $t$ -student distribution is fitted to monthly data returns for CSCO compagny (see Figure 20). The left panel concerns the one with the parameters  $\lambda = -1.24$ ,  $\alpha = 0.014$ ,  $\beta = -0.014$ ,  $\delta = 0.076$ ,  $\mu = 1$  fitted to minimum values. The right panel concerns the one with the parameters  $\lambda = -3.82$ ,  $\alpha = 4.22$ ,  $\beta = 4.22$ ,  $\delta = 0.613$ ,  $\mu = 0.753$  fitted to volume values.

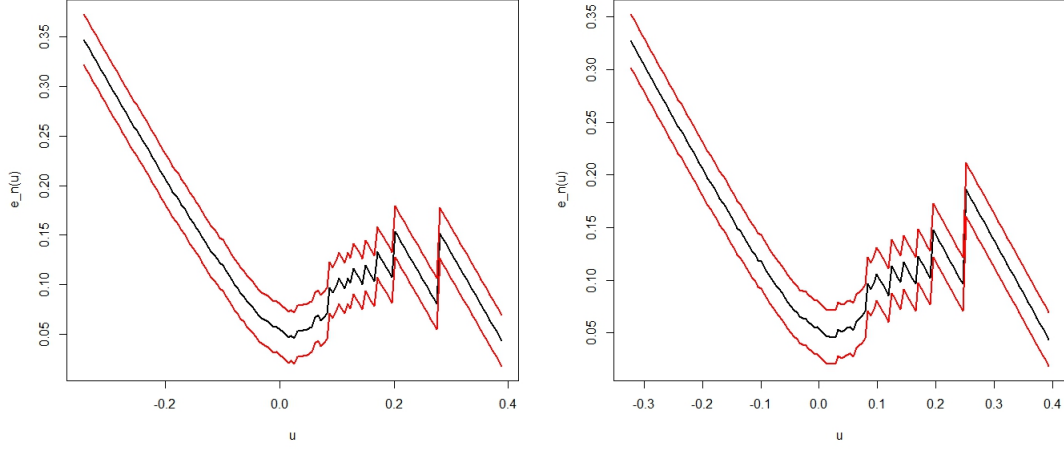


FIGURE 22. *Emef* for CSCO compagny (data log-returns). The left panel concerns monthly opening values and the right panel concerns monthly maximum values.

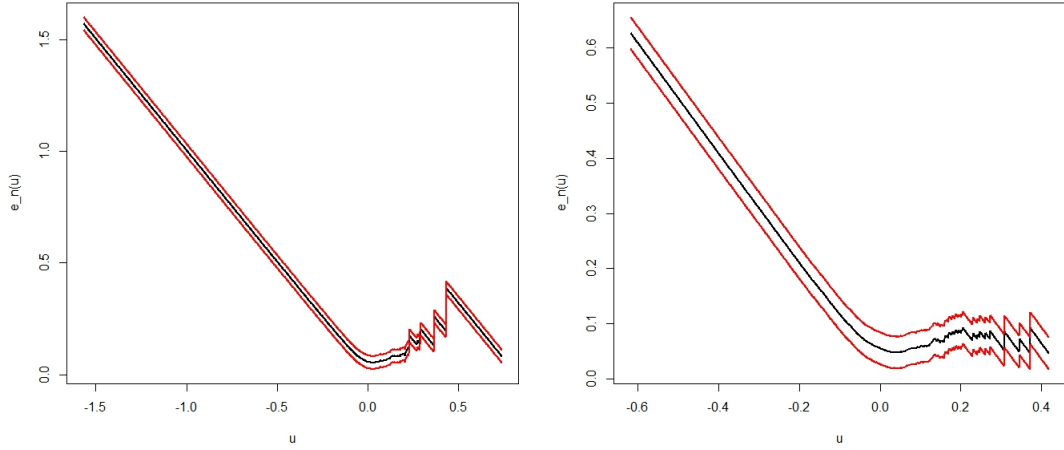


FIGURE 23. A *t*-student distribution is fitted to monthly data log-returns for CSCO compagny (see Figure 22). The left panel concerns the one with the parameters  $\lambda = -1.26$ ,  $\alpha = 0.83$ ,  $\beta = -0.83$ ,  $\delta = 0.07$ ,  $\mu = 0$  fitted to opening values. The right panel concerns the one with the parameters  $\lambda = -1.32$ ,  $\alpha = 0.85$ ,  $\beta = -0.85$ ,  $\delta = 0.076$ ,  $\mu = 0$  fitted to maximum values.

## 6. CONCLUSION

In this paper we have established an asymptotic confidence bands for the mean excess function by using functional process approach. Then we applied these bands for fitting  $Gh$  distributions to Dowjones financial data. It is a known fact that these ones fit well financial data since they embrace major part of classic distributions.

We remarked that Student and  $NIG$  distributions are good candidates for fitting returns and log-returns data showing their semi-heavy tails.

## 7. APPENDIX

## 7.1. Moment computations.

Let  $Z_1, \dots, Z_n$ ,  $n$  i.i.d centered random variables defined on the same probability space with common variance  $\mathbb{E}(Z_i^2) = \kappa_1$  and common fourth moment  $\mathbb{E}(Z_i^4) = \kappa_2 > 0$ . We have

$$\begin{aligned}
 \mathbb{E}[T(n, u, \delta)]^4 &= \mathbb{E}\left(Z_1 + Z_2 + \dots Z_n\right)^4 \\
 &= \mathbb{E}\left(\sum_{k=1}^n Z_k^4 + 6 \sum_{1 \leq i < j \leq n} Z_i^2 Z_j^2\right) \\
 (7.1) \qquad &= \sum_{k=1}^n \mathbb{E}(Z_k^4) + 6 \sum_{1 \leq i < j \leq n} \mathbb{E}(Z_i^2 Z_j^2)
 \end{aligned}$$

since, for distinct  $i, j, k$  and  $l$ ,

$$\mathbb{E}(Z_i Z_j^3) = \mathbb{E}(Z_i Z_j^2 Z_k) = \mathbb{E}(Z_i Z_j Z_k Z_l) = 0,$$

by using independence plus the fact that  $\mathbb{E}(Z_i) = 0$ .

Using independence again,

$$\mathbb{E}(Z_i^2 Z_j^2) = \mathbb{E}(Z_i^2) \mathbb{E}(Z_j^2) = \kappa_1^2 \text{ for } i \neq j.$$

We obtain  $\sum_{1 \leq i < j \leq n} \mathbb{E}(Z_i^2 Z_j^2) = \frac{n(n-1)}{2} \kappa_1^2$  since the number of possible couples  $(i, j)$  of integers such that

$1 \leq i < j \leq n$ , is  $\binom{n}{2} = \frac{n(n-1)}{2}$ . Hence from (7.1), we deduce that

$$\mathbb{E}[T(n, u, \delta)]^4 = n\kappa_2 + 3n(n-1)\kappa_1^2.$$

□

## 7.2. Proofs of the uniform asymptotic consistency bounds.

**7.2.1. Talagrand bounds.** We begin to recall the Talagrand bounds and a device of Einmahl and Mason on how to apply it.

Before going any further, we recall that a class of measurable real valued functions  $\mathcal{F}$  is said to be a *pointwise measurable class* if there exists a countable subclass  $\mathcal{F}_0$  of  $\mathcal{F}$  such as, for any function  $f$  in  $\mathcal{F}$ , we can find a sequence of functions  $\{f_m\}_{m \geq 0}$  in  $\mathcal{F}_0$  for which  $f_m(x) \rightarrow f(x)$  as  $m \rightarrow \infty$ ,  $x \in \mathbb{R}$ . (See Example 2.3.4 in [10].)

Further, let  $\xi_1, \xi_2, \dots$  be a sequence of independent Rademacher random variables independent of  $X_1, X_2, \dots$ , and  $\mathbb{G}_m$  be the functional empirical process indexed by the class of functions  $\mathcal{F}$ .

The following inequality is essentially due to Talagrand (1994) (see [8]).

**Inequality.** Let  $\mathcal{F}$  be a *pointwise measurable class* of functions satisfying for some  $0 < M < \infty$ ,  $\|f\|_\infty \leq M$ ,  $f \in \mathcal{F}$ , .  
Then for all  $t > 0$  we have,

$$(7.2) \quad \begin{aligned} \mathbb{P}\left\{\max_{1 \leq m \leq n} \|\sqrt{m}G_m\|_{\mathcal{F}} \geq A_1 \left(\mathbb{E}\left\|\sum_{i=1}^n \xi_i f(X_i)\right\|_{\mathcal{F}} + t\right)\right\} \\ \leq 2 \left(\exp(-A_2 t^2 / n \sigma_{\mathcal{F}}^2) + \exp(-A_2 t / M)\right), \end{aligned}$$

where  $\sigma_{\mathcal{F}}^2 = \sup_{f \in \mathcal{F}} \text{Var}(f(X))$  and  $A_1, A_2$  are universal constants.

And the lemma below of Einmahl and Mason [9] is very helpful for obtaining bounds on this quantity, when the class  $\mathcal{F}$  has a polynomial covering number.

Assume that there exists a finite valued measurable function  $G$ , called an envelope function, which satisfies for all  $x \in \mathbb{R}$ ,  $G(x) \geq \sup_{f \in \mathcal{F}} |f(x)|$ . We define for  $0 < \epsilon < 1$

$$N(\epsilon, \mathcal{F}) := \sup_Q N\left(\epsilon \sqrt{Q(G^2)}, \mathcal{F}, d_Q\right)$$

where the supremum is taken over all probability measures  $Q$  on  $\mathbb{R}$  for which  $0 < Q(G^2) := \int G^2(y)Q(dy) < \infty$  and  $d_Q$  is the  $L_2(Q)$ -metric. As usual,  $N(\epsilon, \mathcal{F}, d_Q)$  is the minimal number of balls  $\{g : d_Q(g, f) < \epsilon\}$  of  $d_Q$ -radius  $\epsilon$  needed to cover  $\mathcal{F}$ . Here is the device of Einmahl and Mason [9].

**Lemma 3.** (Einmahl - Mason [9]) *Let  $\mathcal{F}$  be a pointwise measurable class of bounded functions such that for some constants  $\beta > 0$ ,  $\nu > 0$ ,  $C > 1$ ,  $\sigma \leq 1/(8C)$  and function  $G$  as above, the following four conditions hold:*

- (A.1)  $\mathbb{E}[G^2(X)] \leq \beta^2$ ;
- (A.2)  $N(\epsilon, \mathcal{F}) \leq C\epsilon^{-\nu}$ ,  $0 < \epsilon < 1$ ;
- (A.3)  $\sigma_0^2 := \sup_{f \in \mathcal{F}} \mathbb{E}[f^2(X)] \leq \sigma^2$ ;
- (A.4)  $\sup_{f \in \mathcal{F}} \|f\|_\infty \leq \frac{1}{2\sqrt{\nu+1}} \sqrt{n\sigma^2 / \log(\beta \vee 1/\sigma)}$ .

Then we have for some absolute constant  $A$ ,

$$(7.3) \quad \mathbb{E}\left\|\sum_{i=1}^n \xi_i f(X_i)\right\|_{\mathcal{F}} \leq A \sqrt{\nu n \sigma^2 \log(\beta \vee 1/\sigma)}.$$

**7.2.2. APPLICATION.** Put  $\ell_u(x) = \ell(x)\mathbb{I}_{(x>u)}$ , with  $\ell(x) = 1$  or  $\ell(x) = x$ , and let  $\mathcal{F} = \{\ell_u, u \in I\}$ .

$\mathcal{F}$  is pointwise measurable since it suffices to take  $\mathcal{F}_0 = \{\ell_u, u \in I \cap \mathbb{Q}\}$ , where  $\mathbb{Q}$  is the set of irrational numbers.

Next  $G = \max(|\ell(u_0)|, |\ell(u_1)|) = M > 0$  is an envelope of  $\mathcal{F}$  since we have

$$\sup_{u \in I} |\ell_u(x)| \leq |\ell(x)| \leq \max(|\ell(u_0)|, |\ell(u_1)|), \quad \forall u_0 \leq x \leq u_1.$$

Remark that if  $\ell(x) = 1$  then  $M = 1$  and if  $\ell(x) = x$  then  $M = \max(|u_0|, |u_1|)$ .

We have  $\sigma_{\mathcal{F}}^2 = \sup_{f \in \mathcal{F}} \text{Var}(f(X)) \leq M^2$ .

So we may use Talagrand's inequality : it remains to check points of **Lemma 3**:

Points (A.1) and (A.3) are obvious with  $\beta = M = \sigma$ .

To check (A.2), consider any probability  $Q$  on  $\mathbb{R}$ . We get for  $(u, v) \in I^2, u \leq v$ ,

$$(7.4) \quad d_Q^2(\ell_u, \ell_v) = \int (\ell_u - \ell_v)^2(x) dQ(x) \leq M^2 Q([u, v]).$$

By a classical result in probability in  $\mathbb{R}$ , for any given  $0 < \varepsilon < 1$ , we may cover  $[u_0, u_1]$  by at most  $m = \lceil \frac{u_1 - u_0}{\varepsilon} \rceil$  sub-intervals  $[s_{i-1}, s_i]$  such that  $Q([s_{i-1}, s_i]) < \varepsilon^2$ ,  $i = 1, \dots, m$ .

( $\lceil x \rceil$  stands for the smallest positive integer greater than or equal to  $x$ ).

Let  $C = (m + 1)\varepsilon$ , we have  $m < C\varepsilon^{-1}$ .

For any  $u \in [u_0, u_1]$ , there exists  $i \in \{1, \dots, m\}$  such as  $s_{i-1} \leq u \leq s_i$  with  $Q([u, s_i]) < \varepsilon^2$ , so the corresponding  $\ell_u \in \mathcal{F}$  is such that

$$d_Q(\ell_u, \ell_{s_i}) < \varepsilon M = \varepsilon \sqrt{Q(G^2)} \quad \text{from (7.4).}$$

To finish  $m = N(\varepsilon \sqrt{Q(G^2)}, \mathcal{F}, d_Q) < C\varepsilon^{-1}$  and

$$N(\varepsilon, \mathcal{F}) = \sup_Q N(\varepsilon \sqrt{Q(G^2)}, \mathcal{F}, d_Q) \leq C\varepsilon^{-1}.$$

Now we take  $\beta^2 = \sigma^2 = \max(2, \max(|\ell(u_0)|, |\ell(u_1)|)) = M_1$ .

Finally for

$$n \geq \frac{8M^2 \log M_1}{M_1^2},$$

we have

$$(7.5) \quad \mathbb{E} \left\| \sum_{i=1}^n \xi_i g(X_i) \right\|_{\mathcal{F}} \leq C_{\mathcal{F}} \sqrt{n},$$

where  $C_{\mathcal{F}} = A M_1 \sqrt{\log M_1}$ , since all the points of the **Lemma 3** are checked.

Now we are going to apply the inequality (7.2) first for the class of functions

$$\mathcal{F}_1 = \{\ell_u(x) = g_u(x), u \in I\}.$$

In this case  $M_1 = 2$  since  $\ell(x) = 1$ , for any  $u_0 \leq x \leq u_1$ , and

$$\mathbb{E} \left\| \sum_{i=1}^n \xi_i g_u(X_i) \right\|_{\mathcal{F}_1} = D_{n,1} \leq C_{\mathcal{F}_1} \sqrt{n}, \quad \text{where } C_{\mathcal{F}_1} = 2A \sqrt{\log 2}.$$

Let  $\varepsilon > 0$ ,  $n_1 \geq 2 \log 2$  and  $t_0$  such that

$$\exp\left(-\frac{A_2 t_0^2}{n_1}\right) \leq \frac{\varepsilon}{8}, \quad \text{and} \quad \exp\left(-A_2 t_0\right) \leq \frac{\varepsilon}{8} \quad \text{and} \quad t_0 < \sqrt{n_1}.$$

(Remind that  $\sigma_{\mathcal{F}}^2 = 1$ .)

Then

$$\mathbb{P}\left\{ \max_{1 \leq m \leq n} \|\sqrt{m} \mathbb{G}_m\|_{\mathcal{F}_1} \geq A_1 \left( \mathbb{E} \left\| \sum_{i=1}^n \xi_i g_u(X_i) \right\|_{\mathcal{F}_1} + t_0 \right) \right\} \leq \varepsilon/2.$$

So for  $n \geq n_1$ , we arrive at

$$\mathbb{P}\left( |\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)| < \frac{A_1(D_{n,1} + t_0)}{n}, u \in I \right) > 1 - \varepsilon/2.$$

As  $t_0/\sqrt{n} < \sqrt{n_1}/\sqrt{n} \leq 1$ , we obtain

$$\frac{A_1(D_{n,1} + t_0)}{n} \leq \frac{A_1 C_{\mathcal{F}_1} \sqrt{n} + A_1 \sqrt{n}}{n} = \frac{A_1 C_{\mathcal{F}_1} + A_1}{\sqrt{n}} = \frac{D_1}{\sqrt{n}},$$

thus

$$(7.6) \quad \mathbb{P}\left( |\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)| < \frac{D_1}{\sqrt{n}}, u \in I \right) > 1 - \varepsilon/2$$

where  $D_1 = 2A A_1 \sqrt{\log 2} + A_1$ .

Let us use the same method, for the class of functions

$$\mathcal{F}_2 = \{\ell_u(x) = f_u(x), u \in I\}.$$

In this case  $M_1 = \max(2, \max(|u_0|, |u_1|))$  since  $\ell(x) = x$ , for any  $u_0 \leq x \leq u_1$ , and

$$\mathbb{E} \left\| \sum_{i=1}^n \xi_i f_u(X_i) \right\|_{\mathcal{F}_2} = D_{n,2} \leq C_{\mathcal{F}_2} \sqrt{n}, \text{ where } C_{\mathcal{F}_2} = AM_1 \sqrt{\log M_1}.$$

Let  $n_2 \geq \frac{8M^2 \log M_1}{M_1^2}$  and  $t_0$  such that

$$\exp\left(\frac{-A_2 t_0^2}{n_2}\right) \leq \frac{\varepsilon}{8} \quad \text{and} \quad \exp\left(-A_2 t_0\right) \leq \frac{\varepsilon}{8} \quad \text{and} \quad t_0 < \sqrt{n_2}.$$

Then

$$\mathbb{P}\left\{ \max_{1 \leq m \leq n} \|\sqrt{m} G_m\|_{\mathcal{F}_2} \geq A_1 \left( \mathbb{E} \left\| \sum_{i=1}^n \xi_i f_u(X_i) \right\|_{\mathcal{F}_2} + t_0 \right) \right\} \leq \varepsilon/2.$$

So for  $n \geq n_2$ , we deduce that

$$\mathbb{P}\left( |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| < \frac{A_1(D_{n,2} + t_0)}{n}, u \in I \right) > 1 - \varepsilon/2.$$

As  $t_0/\sqrt{n} < \sqrt{n_1}/\sqrt{n} \leq 1$ , we obtain

$$\frac{A_1(D_{n,2} + t_0)}{n} \leq \frac{A_1 C_{\mathcal{F}_2} \sqrt{n} + A_1 \sqrt{n}}{n} = \frac{A_1 C_{\mathcal{F}_2} + A_1}{\sqrt{n}} = \frac{D_2}{\sqrt{n}},$$

thus

$$(7.7) \quad \mathbb{P}\left( |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| < \frac{D_2}{\sqrt{n}}, u \in I \right) > 1 - \varepsilon/2,$$

where  $D_2 = A A_1 M_1 \sqrt{\log M_1} + A_1$ .

Now we use again (2.3) :

$$|e_n(u) - e(u)| \leq |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| \times |\mathbb{P}_n(g_u)|^{-1} + |\mathbb{P}_X(f_u)| \times \frac{|\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)|}{|\mathbb{P}_n(g_u) \mathbb{P}_X(g_u)|}.$$

For  $u_0 \leq u \leq u_1$ , we get

$$0 < \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \leq \mathbb{P}_X(g_u) - \frac{D_1}{\sqrt{n}} < \mathbb{P}_n(g_u) < \mathbb{P}_X(g_u) + \frac{D_1}{\sqrt{n}}$$

with a probability greater than  $(w.p.g.t) 1 - \varepsilon/2$  and thus

$$|\mathbb{P}_n(g_u)|^{-1} < \left( \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \right)^{-1} \quad w.p.g.t \quad 1 - \varepsilon/2$$

so we obtain

$$\begin{aligned} \mathbb{P}\left( |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| \times |\mathbb{P}_n(g_u)|^{-1} < \frac{D_2}{\sqrt{n}} \times \left( \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \right)^{-1}, u \in I \right) \\ \geq \mathbb{P}\left( |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| < \frac{D_2}{\sqrt{n}}, u \in I \right) > 1 - \varepsilon/2, \end{aligned}$$

thus

$$(7.8) \quad \mathbb{P}\left( |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| \times |\mathbb{P}_n(g_u)|^{-1} < \frac{D_2}{\sqrt{n}} \times \left( \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \right)^{-1}, u \in I \right) > 1 - \varepsilon/2.$$



From the following inequalities :

$$\begin{cases} |\mathbb{P}_n(g_u)|^{-1} \leq \left( \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \right)^{-1}, & w.p.g.t \ 1 - \varepsilon/2, \\ |\mathbb{P}_X(g_u)|^{-1} \leq \bar{F}(u_1)^{-1}, \\ |\mathbb{P}_X(f_u)| \leq \mathbb{E}|X|, \end{cases}$$

and by the same manner, we obtain

$$\begin{aligned} \mathbb{P} \left( |\mathbb{P}_X(f_u)| \times \frac{|\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)|}{|\mathbb{P}_n(g_u)\mathbb{P}_X(g_u)|} < \frac{D_1}{\sqrt{n}} \times \mathbb{E}|X| \times \left( \bar{F}(u_1) \left( \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \right) \right)^{-1}, u \in I \right) \\ \geq \mathbb{P} \left( |\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)| < \frac{D_1}{\sqrt{n}}, u \in I \right) > 1 - \varepsilon/2, \end{aligned}$$

thus

$$(7.9) \quad \mathbb{P} \left( |\mathbb{P}_X(f_u)| \times \frac{|\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)|}{|\mathbb{P}_n(g_u)\mathbb{P}_X(g_u)|} < \frac{D_1}{\sqrt{n}} \times \mathbb{E}|X| \times \left( \bar{F}(u_1) \left( \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \right) \right)^{-1} \right) > 1 - \varepsilon/2$$

By combining (7.8) and (7.9), we obtain

$$\begin{aligned} \mathbb{P} \left( |\mathbb{P}_n(g_u)|^{-1} \times |\mathbb{P}_n(f_u) - \mathbb{P}_X(f_u)| \geq \frac{D_2}{\sqrt{n}} \times \left( \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \right)^{-1}, u \in I \right) \\ + \mathbb{P} \left( |\mathbb{P}_X(f_u)| \times \frac{|\mathbb{P}_n(g_u) - \mathbb{P}_X(g_u)|}{|\mathbb{P}_n(g_u)\mathbb{P}_X(g_u)|} \geq \frac{D_1}{\sqrt{n}} \times \mathbb{E}|X| \times \left( \bar{F}(u_1) \left( \bar{F}(u_1) - \frac{D_1}{\sqrt{n}} \right) \right)^{-1} \right) \\ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

This gives

$$\mathbb{P}(|e_n(u) - e(u)| \geq \frac{E_n}{\sqrt{n}}) \leq \varepsilon \quad \text{where} \quad E_n = \frac{1}{\bar{F}(u_1) - D_1/\sqrt{n}} \left( D_2 + \frac{D_1 \times \mathbb{E}|X|}{\bar{F}(u_1)} \right).$$

Finally we conclude by :

$$\mathbb{P} \left( e_n(u) - \frac{E_n}{\sqrt{n}} < e(u) < e_n(u) + \frac{E_n}{\sqrt{n}}, u \in I \right) > 1 - \varepsilon$$

for any  $\varepsilon > 0$ , any  $n \geq n_0$  with

$$E_n = \frac{1}{\bar{F}(u_1) - D_1/\sqrt{n}} \left( D_2 + \frac{D_1 \times \mathbb{E}|X|}{\bar{F}(u_1)} \right) \quad \text{and} \quad \begin{cases} D_1 = 2AA_1\sqrt{\log 2} + A_1 \\ D_2 = A A_1 M_1 \sqrt{\log M_1} + A_1, \\ M_1 = \max(2, \max(|u_0|, |u_1|)) \end{cases}$$

□

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